

Line & Surface Integral Notation

PARAMETRIZED CURVES & LINE INTEGRALS:

Curve: (Position)

$\vec{r}(t) = (x(t), y(t), z(t))$

Tangent Vector: (Velocity)

$$\vec{v} = \frac{d\vec{r}}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$$

Tangent Vector Differential or Vector Differential of Arc Length:

$$\vec{ds} = \vec{dr} = (dx, dy, dz) = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) dt = \vec{v}\, dt = \hat{v}|\vec{v}|\, dt = \hat{v}\, ds$$

Tangent Scalar Differential or Scalar Differential of Arc Length:

$$ds = |\vec{ds}| = \sqrt{(dx)^2 + (dy)^2 + (dz)^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = |\vec{v}|\, dt$$

Arc Length Integral:

$$L = \int_A^B ds = \int_a^b |\vec{v}|\, dt$$

Integral of a Scalar Function $f(x,y,z)$ along $\vec{r}(t)$ from $A = \vec{r}(a)$ to $B = \vec{r}(b)$:

$$\int_A^B f\, ds = \int_a^b f(\vec{r}(t))\, |\vec{v}|\, dt$$

Average Value of a Function $f(x,y,z)$ along $\vec{r}(t)$ from $A = \vec{r}(a)$ to $B = \vec{r}(b)$:

$$f_{\text{ave}} = \frac{1}{L}\int_A^B f\, ds = \frac{1}{L}\int_a^b f(\vec{r}(t))\, |\vec{v}|\, dt$$

Total Mass:

$$M = \int_A^B \rho\, ds = \int_a^b \rho\, |\vec{v}|\, dt$$

Center of Mass:

$$(\bar{x}, \bar{y}, \bar{z}) = \frac{1}{M}\int_A^B (x,y,z)\, \rho\, ds$$

Integral of a Vector Field $\vec{F} = (F_1, F_2, F_3)$ along $\vec{r}(t)$ from $A = \vec{r}(a)$ to $B = \vec{r}(b)$: (Work or Circulation)

$$\int_A^B \vec{F} \cdot \vec{ds} = \int_A^B (F_1\, dx + F_2\, dy + F_3\, dz) = \int_a^b \left(F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt}\right) dt = \int_a^b \vec{F} \cdot \vec{v}\, dt = \int_A^B \vec{F} \cdot \hat{v}\, ds$$

SPECIAL FOR CURVES IN R^2:

Normal Vector:

$$\vec{n} = \vec{v}^{\perp} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} \\ v_1 & v_2 \end{vmatrix} = v_2 \hat{\imath} - v_1 \hat{\jmath} = \frac{dy}{dt}\hat{\imath} - \frac{dx}{dt}\hat{\jmath} \qquad |\vec{n}| = |\vec{v}|$$

Normal Vector Differential:

$$\vec{dn} = dy\, \hat{\imath} - dx\, \hat{\jmath} = \left(\frac{dy}{dt}\hat{\imath} - \frac{dx}{dt}\hat{\jmath}\right) dt = \vec{n}\, dt = \hat{n}|\vec{n}|\, dt = \hat{n}\, ds$$

Normal Scalar Differential:

$$dn = |\vec{dn}| = \sqrt{(dy)^2 + (dx)^2} = ds$$

Integral of the Normal Component of Vector Field $\vec{G} = G_1 \hat{\imath} + G_2 \hat{\jmath}$ along $\vec{r}(t)$:

$$\int_A^B \vec{G} \cdot \vec{dn} = \int_A^B (G_1 dy - G_2 dx) = \int_a^b \left(G_1 \frac{dy}{dt} - G_2 \frac{dx}{dt}\right) dt = \int_a^b \vec{G} \cdot \vec{n}\, dt = \int_A^B \vec{G} \cdot \hat{n}\, ds$$

Further, if $\vec{G} = \vec{F}^{\perp} = F_2 \hat{\imath} - F_1 \hat{\jmath}$ then:

$$\vec{G} \cdot \vec{n} = (F_2, -F_1) \cdot (v_2, -v_1) = \vec{F} \cdot \vec{v}$$

and

$$\int_A^B \vec{G} \cdot \vec{dn} = \int_A^B (F_2\, dy - (-F_1)\, dx) = \int_A^B \vec{F} \cdot \vec{ds}$$

PARAMETRIZED SURFACES & SURFACE INTEGRALS:

Surface: (Position)

$$\vec{R}(u,v) = (x(u,v), y(u,v), z(u,v))$$

Tangent Vectors:

$$\vec{e}_u = \frac{\partial \vec{R}}{\partial u} = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right) \quad \text{and} \quad \vec{e}_v = \frac{\partial \vec{R}}{\partial v} = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right)$$

Normal Vector:

$$\vec{N} = \vec{e}_u \times \vec{e}_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} = \frac{\partial(y,z)}{\partial(u,v)}\hat{\imath} + \frac{\partial(z,x)}{\partial(u,v)}\hat{\jmath} + \frac{\partial(x,y)}{\partial(u,v)}\hat{k}$$

Normal Vector Differential or Vector Differential of Surface Area:

$$d\vec{S} = (dy\,dz, dz\,dx, dx\,dy) = \left(\frac{\partial(y,z)}{\partial(u,v)}, \frac{\partial(z,x)}{\partial(u,v)}, \frac{\partial(x,y)}{\partial(u,v)} \right) du\,dv$$

$$= \vec{N}\,du\,dv = \hat{N}|\vec{N}|\,du\,dv = \hat{N}\,dS$$

Normal Scalar Differential or Scalar Differential of Surface Area:

$$dS = \left|d\vec{S}\right| = \sqrt{(dy\,dz)^2 + (dz\,dx)^2 + (dx\,dy)^2} = \sqrt{\left(\frac{\partial(y,z)}{\partial(u,v)} \right)^2 + \left(\frac{\partial(z,x)}{\partial(u,v)} \right)^2 + \left(\frac{\partial(x,y)}{\partial(u,v)} \right)^2}\,du\,dv$$

$$= \left|\vec{N}\right| du\,dv$$

Surface Area Integral:

$$A = \iint_{\vec{R}} dS = \iint_{\vec{R}} \left|\vec{N}\right| du\,dv$$

Integral of a Scalar Function $f(x,y,z)$ over $\vec{R}(u,v)$:

$$\iint_{\vec{R}} f\,dS = \iint_{\vec{R}} f\left(\vec{R}(u,v)\right)\left|\vec{N}\right| du\,dv$$

Average Value of a Function $f(x,y,z)$ over $\vec{R}(u,v)$:

$$f_{\text{ave}} = \frac{1}{A}\iint_{\vec{R}} f\,dS = \frac{1}{A}\iint_{\vec{R}} f\left(\vec{R}(u,v)\right)\left|\vec{N}\right| du\,dv$$

Total Mass:

$$M = \iint_{\vec{R}} \rho\,dS = \iint_{\vec{R}} \rho\left|\vec{N}\right| du\,dv$$

Center of Mass:

$$(\bar{x}, \bar{y}, \bar{z}) = \frac{1}{M}\iint_{\vec{R}} (x,y,z)\,\rho\,dS$$

Integral of a VecField $\vec{F} = (F_1, F_2, F_3)$ over $\vec{R}(u,v)$: (Flux or Expansion)

$$\iint_{\vec{R}} \vec{F} \cdot d\vec{S} = \iint_{\vec{R}} (F_1\,dy\,dz + F_2\,dz\,dx + F_3\,dx\,dy) = \iint_{\vec{R}} \left(F_1 \frac{\partial(y,z)}{\partial(u,v)} + F_2 \frac{\partial(z,x)}{\partial(u,v)} + F_3 \frac{\partial(x,y)}{\partial(u,v)} \right) du\,dv$$

$$= \iint_{\vec{R}} \vec{F} \cdot \vec{N}\,du\,dv = \iint_{\vec{R}} \vec{F} \cdot \hat{N}\,dS$$