

Definition and Properties of a Vector Space

Definition:

A Vector Space is a set V with the operations of vector addition $\oplus$ and scalar multiplication $\odot$ satisfying a set of axioms.

$\oplus : V \times V \to V : (u,v) \in V \times V \mapsto u \oplus v \in V$

$\odot : \mathbb{R} \times V \to V : (c,v) \in \mathbb{R} \times V \mapsto c \odot v \in V$

Axioms:

A1: $u \oplus v = v \oplus u$ Addition is commutative

A2: $(u \oplus v) \oplus w = u \oplus (v \oplus w)$ Addition is associative

A3: $\exists \mathbf{0} \in V$ such that $v \oplus \mathbf{0} = v$ Existance of a zero

A4: $\forall v \exists \ominus v$ such that $v \oplus \ominus v = \mathbf{0}$ Existance of negatives

A5: $c \odot (u \oplus v) = c \odot u \oplus c \odot v$ Scalar multiplication distributes over vector addition

A6: $(c + d) \odot v = c \odot v \oplus d \odot v$ Scalar multiplication distributes over scalar addition

A7: $(cd) \odot v = c \odot (d \odot v)$ Scalar multiplication is associative.

A8: $1 \odot v = v$ 1 is the identity for scalar multiplication.

Properties:

P1: $0 \odot v = \mathbf{0}$

P2: $x \oplus y = \mathbf{0} \quad \Rightarrow \quad y = \ominus x$

P3: $(-1) \odot v = \ominus v$

P4: $c \odot \mathbf{0} = \mathbf{0}$

P5: $c \odot v = \mathbf{0} \quad \Rightarrow \quad$ either $c = 0$ or $v = \mathbf{0}$