Name____________________________ ID______________

1	/30	4	/20
2	/10	5	/10
3	/15	6	/15

MATH 311 Final Exam Spring 2001
Section 200 Solutions P. Yasskin

1. $\left(30\text{ points}\right)$ Let $S(2,2)$ be the set of 2×2 symmetric matrices, i.e. 2×2 matrices M satisfying $M^{\mathrm{T}} = M$. Consider the function $L : M(2,2) \to S(2,2)$ given by $L(X) = X + X^{\mathrm{T}}$.

a. (5) Show that $S(2,2)$ is a subspace of $M(2,2)$, the vector space of 2×2 matrices.

$$A,B \in S \quad\Rightarrow\quad A^{\mathrm{T}} = A,\ B^{\mathrm{T}} = B \quad\Rightarrow\quad (sA+tB)^{\mathrm{T}} = sA^{\mathrm{T}} + tB^{\mathrm{T}} = sA + tB \quad\Rightarrow\quad sA+tB \in S$$

b. (5) Find a basis for $S(2,2)$. What is the dimension of $S(2,2)$?

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in S \quad\Leftrightarrow\quad b = c$$

$$M = \begin{pmatrix} a & b \\ b & d \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Basis is $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Dimension $= 3$

c. (5) Show L is linear.

Recall $L(X) = X + X^{\mathrm{T}}$. So

$$L(sA+tB) = (sA+tB) + (sA+tB)^{\mathrm{T}} = (sA+tB) + (sA^{\mathrm{T}} + tB^{\mathrm{T}}) = s(A+A^{\mathrm{T}}) + t(B+B^{\mathrm{T}}) = sL(A) + tL(B)$$

d. (15) For the linear function L, identify

- (1) $Dom(L) = M(2,2)$

  $\dim Dom(L) = 4$

- (1) $CoDom(L) = S(2,2)$

  $\dim CoDom(L) = 3$

- (3) $Ker(L) = \left\{M \text{ s.t. } L(M) = 0\right\} = \left\{M \text{ s.t. } M + M^{\mathrm{T}} = 0\right\} = \left\{M \text{ s.t. } M^{\mathrm{T}} = -M\right\}$

  $=$ antisymmetric matrices

  $$= \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ s.t. } \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}\right\} = \left\{\begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}\right\}$$

  $$= Span\left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\}$$

  $\dim Ker(L) = 1$

- (3) $Ran(L) = \{L(M)\} = \{M + M^{\mathsf{T}}\} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a & c \\ b & d \end{pmatrix} \right\} = \left\{ \begin{pmatrix} 2a & b+c \\ b+c & 2d \end{pmatrix} \right\}$

  $= \left\{ 2a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + (b+c)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 2d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

  $= Span\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} = S(2,2)$

  $\dim Ran(L) = 3$

- (3) $1-1$? Circle: Yes [✓ No]

  Why? $\dim Ker(L) = 1 \neq 0$

- (3) *onto*? Circle: [✓ Yes] No

  Why? $Ran(L) = CoDom(L) = S(2,2)$

- (1) Verify the Nullity-Rank Theorem for L.

  $\dim Ker(L) + \dim Ran(L) = 1 + 3 = 4 = \dim M(2,2) = \dim Dom(L)$

2. (10 points) Consider the function of two matrices $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $Y = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ given by

$$\langle X, Y \rangle = tr(XY)$$

where tr means "trace" which is the sum of the principle diagonal entries, i.e. $tr\begin{pmatrix} w & x \\ y & z \end{pmatrix} = w + z$. Explain why $\langle\ ,\ \rangle$ is an inner product on $S(2,2)$, but is not an inner product on $M(2,2)$.

$$\langle X, Y \rangle = tr(XY) = tr\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix} \right) = tr\begin{pmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{pmatrix} = ap + br + cq + ds$$

$$\langle Y, X \rangle = tr(YX) = tr\left(\begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = tr\begin{pmatrix} ap+cq & pb+qd \\ ra+sc & br+ds \end{pmatrix} = ap + cq + br + ds = \langle X, Y \rangle$$

So $\langle\ ,\ \rangle$ is symmetric.

$$\langle rX + Z, Y \rangle = tr((rX+Z)Y) = tr(rXY + ZY) = tr(rXY) + tr(ZY) = r \bullet tr(XY) + tr(ZY) = r \bullet \langle X, Y \rangle + \langle Z, Y \rangle$$

So $\langle\ ,\ \rangle$ is linear.

$$\langle X, X \rangle = tr(XX) = tr\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = tr\begin{pmatrix} a^2+bc & ab+bd \\ ca+dc & bc+d^2 \end{pmatrix} = a^2 + 2bc + d^2$$

If $X \in S(2,2)$ then $b = c$ and $\langle X, X \rangle = a^2 + 2b^2 + d^2 \geq 0$ and $= 0$ only if $a = b = d = 0$, so that $X = 0$. So $\langle\ ,\ \rangle$ is positive definite on $S(2,2)$.

If $X \in M(2,2)$ then $a^2 + 2bc + d^2$ may not be positive, e.g. if $X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ then $\langle X, X \rangle = -2$.

So $\langle\ ,\ \rangle$ is NOT positive definite on $M(2,2)$.

3. $\big($15 points$\big)$ Consider the linear map $L : P_2 \to \mathbf{R}^3$ given by $L(p) = \begin{pmatrix} p(-1) \\ p(0) \\ p(1) \end{pmatrix}$.

a. (5) Find the matrix of L relative to the bases $e = \{e_1 = 1, e_2 = t, e_3 = t^2\}$ for P_2 and $i = \left\{ \vec{i}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \vec{i}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \vec{i}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ for $\mathbf{R}^3$. Call it A.

$$L(e_1) = \begin{pmatrix} e_1(-1) \\ e_1(0) \\ e_1(1) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \vec{i}_1 + \vec{i}_2 + \vec{i}_3$$

$$L(e_2) = \begin{pmatrix} e_2(-1) \\ e_2(0) \\ e_2(1) \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = -\vec{i}_1 + \vec{i}_3 \qquad \underset{i\leftarrow e}{A} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$$L(e_3) = \begin{pmatrix} e_3(-1) \\ e_3(0) \\ e_3(1) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \vec{i}_1 + \vec{i}_3$$

b. (5) Find the matrix of L relative to the bases $q = \{q_1 = 1 + t^2, q_2 = t + t^2, q_3 = t^2\}$ for P_2 and $v = \left\{ \vec{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \right\}$ for $\mathbf{R}^3$. Call it B.

$$L(q_1) = \begin{pmatrix} q_1(-1) \\ q_1(0) \\ q_1(1) \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \vec{v}_3$$

$$L(q_2) = \begin{pmatrix} q_2(-1) \\ q_2(0) \\ q_2(1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} = 2\vec{v}_1 \qquad \underset{v\leftarrow q}{B} = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$L(q_3) = \begin{pmatrix} q_3(-1) \\ q_3(0) \\ q_3(1) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \vec{v}_2$$

c. (5) Find the matrix B by a second method.

$$\underset{v\leftarrow q}{B} = \underset{v\leftarrow i}{C} \quad \underset{i\leftarrow e}{A} \quad \underset{e\leftarrow q}{C}$$

$$q_1 = 1 + t^2 = \vec{e}_1 + \vec{e}_3, \qquad q_2 = t + t^2 = \vec{e}_2 + \vec{e}_3, \qquad q_3 = t^2 = \vec{e}_3 \qquad \underset{e\leftarrow q}{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\vec{v}_1 = \vec{i}_3, \qquad \vec{v}_2 = \vec{i}_1 + \vec{i}_3, \qquad \vec{v}_3 = 2\vec{i}_1 + \vec{i}_2 + 2\vec{i}_3 \qquad \underset{i\leftarrow v}{C} = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix} \qquad \underset{v\leftarrow i}{C} = \begin{pmatrix} -1 & 0 & 1 \\ 1 & -2 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\underset{v\leftarrow q}{B} = \underset{v\leftarrow i}{C} \quad \underset{i\leftarrow e}{A} \quad \underset{e\leftarrow q}{C} = \begin{pmatrix} -1 & 0 & 1 \\ 1 & -2 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

4. $\big(20$ points$\big)$ Consider the helix H parametrized by $\vec{r}(t) = (4\cos t, 4\sin t, 3t)$ between $A = (4,0,0)$ and $B = (-4,0,3\pi)$.

a. (10) Compute the line integral $\underset{H}{\int_A^B} \vec{F} \cdot d\vec{s}$ of the vector field $\vec{F} = (yz, -xz, z)$ along the helix H.

$$0 \le t \le \pi \qquad \vec{v} = (-4\sin t, 4\cos t, 3) \qquad \vec{F} = (yz, -xz, z) = (12t\sin t, -12t\cos t, 3t)$$

$$\vec{F} \cdot \vec{v} = -48t\sin^2 t - 48t\cos^2 t + 9t = -48t + 9t = -39t$$

$$\underset{H}{\int_A^B} \vec{F} \cdot d\vec{s} = \int_0^\pi -39t\,dt = -39\frac{t^2}{2}\Big|_0^\pi = -\frac{39\pi^2}{2}$$

b. (10) Find the total mass of the helix H if the linear mass density is $\rho = z^2$.

$$|\vec{v}| = \sqrt{16\sin^2 t + 16\cos^2 t + 9} = 5 \qquad \rho = z^2 = 9t^2$$

$$M = \int \rho\,ds = \int \rho\,|\vec{v}|\,dt = \int_0^\pi 9t^2\,5\,dt = 45\frac{t^3}{3}\Big|_0^\pi = 15\pi^3$$

5. $\big(10$ points$\big)$ Compute $\oint x\ dx + z\ dy - y\ dz$ around the boundary of the triangle with vertices $(0,0,0)$, $(0,1,0)$ and $(0,0,1)$, traversed in this order of the vertices.
HINT: The yz-plane may be parametrized as $\vec{R}(u,v) = (0,u,v)$.

By Stokes therorm $\underset{\partial T}{\int} \vec{F} \cdot d\vec{s} = \underset{T}{\iint} \vec{\nabla} \times \vec{F} \cdot d\vec{S}$ where T is the triangle and ∂T is its boundary. From the direction the boundary is traversed, the normal to T must point in the positive x-direction.

$$\vec{e}_u = (0,1,0) \qquad \vec{e}_v = (0,0,1) \qquad \vec{N} = \vec{e}_u \times \vec{e}_v = (1,0,0)$$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ x & z & -y \end{vmatrix} = (-2,0,0) \qquad \vec{\nabla} \times \vec{F} \cdot \vec{N} = -2$$

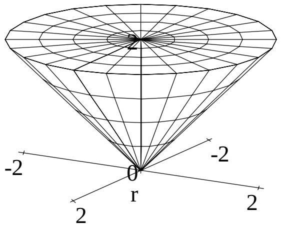
The triangle satisfies $0 \le y \le 1$ and $0 \le z \le 1 - y$, or $0 \le u \le 1$ and $0 \le v \le 1 - u$. So

$$\underset{T}{\iint} \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \int_0^1 \int_0^{1-u} (-2)\,dv\,du = \int_0^1 (-2)(1-u)\,du = -2\Big[u - \frac{u^2}{2}\Big]_0^1 = -1$$

6. $\left(15\right.$ points$\left.\right)$ Gauss' Theorem states

$$\iiint\limits_V \vec{\nabla} \cdot \vec{F}\, dV = \iint\limits_{\partial V} \vec{F} \cdot d\vec{S}$$

where ∂V is the total boundary of V with OUTWARD normal.



Let V be the solid cone $\sqrt{x^2 + y^2} \le z \le 2$.

Let C be the conical surface $z = \sqrt{x^2 + y^2}$ for $z \le 2$

with UPWARD normal.

Let D be the disk $x^2 + y^2 \le 4$ with $z = 2$

with UPWARD normal.

Compute $\iint\limits_C \vec{F} \cdot d\vec{S}$ for $\vec{F} = (xy^2, yx^2, z^3)$ in two ways.

a. (5) Method I: Parametrize C and compute $\iint\limits_C \vec{F} \cdot d\vec{S}$ explicitly.

C: Polar coordinates:

$\vec{R}_C(r,\theta) = (r\cos\theta, r\sin\theta, r)$ $\qquad \vec{F} = (xy^2, yx^2, z^3) = \left(r^3\cos\theta\sin^2\theta, r^3\sin\theta\cos^2\theta, r^3\right)$

$\vec{e}_r = (\cos\theta, \sin\theta, 1)$ $\qquad \vec{e}_\theta = (-r\sin\theta, r\cos\theta, 0)$ $\qquad \vec{N} = \vec{e}_r \times \vec{e}_\theta = (-r\cos\theta, -r\sin\theta, r)$ $\qquad$ UP, correct

$$\iint\limits_D \vec{F} \cdot d\vec{S} = \iint\limits_D \vec{F} \cdot \vec{N}\, dr\, d\theta = \int_0^{2\pi}\int_0^2 \left(-r^4\cos^2\theta\sin^2\theta - r^4\sin^2\theta\cos^2\theta + r^4\right) dr\, d\theta$$

$$= \left[\frac{r^5}{5}\right]_0^2 \int_0^{2\pi} \left(-2\sin^2\theta\cos^2\theta + 1\right) d\theta = \frac{32}{5}\int_0^{2\pi}\left(1 - \frac{1}{2}\sin^2(2\theta)\right) d\theta$$

$$= \frac{32}{5}\left[2\pi - \frac{1}{2}\frac{1}{2}(2\pi)\right] = \frac{32}{5}\left[\frac{3}{2}\pi\right] = \frac{48}{5}\pi$$

b. (10) Method II: Parametrize D and V, compute $\iint\limits_D \vec{F} \cdot d\vec{S}$ and $\iiint\limits_V \vec{\nabla} \cdot \vec{F}\, dV$ and solve for $\iint\limits_C \vec{F} \cdot d\vec{S}$.

Be very careful with the orientation of the surfaces.

D: Polar coordinates:

$\vec{R}_D(r,\theta) = (r\cos\theta, r\sin\theta, 2)$ $\qquad \vec{F} = (xy^2, yx^2, z^3) = \left(r^3\cos\theta\sin^2\theta, r^3\sin\theta\cos^2\theta, 8\right)$

$\vec{e}_r = (\cos\theta, \sin\theta, 0)$ $\qquad \vec{e}_\theta = (-r\sin\theta, r\cos\theta, 0)$ $\qquad \vec{N} = \vec{e}_r \times \vec{e}_\theta = (0, 0, r)$ $\qquad$ UP, correct

$$\iint\limits_D \vec{F} \cdot d\vec{S} = \iint\limits_D \vec{F} \cdot \vec{N}\, dr\, d\theta = \int_0^{2\pi}\int_0^2 8r\, dr\, d\theta = 2\pi\left[4r^2\right]_0^2 = 32\pi$$

V: Cylindrical coordinates:

$\vec{R}_V(r,\theta,z) = (r\cos\theta, r\sin\theta, z)$ $\qquad J = r$ $\qquad \vec{\nabla} \cdot \vec{F} = y^2 + x^2 + 3z^2 = r^2 + 3z^2$

$$\iiint\limits_V \vec{\nabla} \cdot \vec{F}\, dV = \int_0^2\int_0^{2\pi}\int_r^2 (r^2 + 3z^2)\, r\, dz\, d\theta\, dr = 2\pi\int_0^2 \left[r^2 z + z^3\right]_r^2 r\, dr$$

$$= 2\pi\int_0^2 ([r^2 2 + 8] - [r^3 + r^3])\, r\, dr = 2\pi\int_0^2 (2r^3 + 8r - 2r^4)\, dr = 2\pi\left[\frac{1}{2}r^4 + 4r^2 - \frac{2}{5}r^5\right]_0^2$$

$$= 2\pi\left(8 + 16 - \frac{64}{5}\right) = \frac{112}{5}\pi$$

Since C is oriented upward, $\iiint\limits_V \vec{\nabla} \cdot \vec{F}\, dV = \iint\limits_D \vec{F} \cdot d\vec{S} - \iint\limits_C \vec{F} \cdot d\vec{S}$

So $\iint\limits_C \vec{F} \cdot d\vec{S} = \iint\limits_D \vec{F} \cdot d\vec{S} - \iiint\limits_V \vec{\nabla} \cdot \vec{F}\, dV = 32\pi - \frac{112}{5}\pi = \frac{48}{5}\pi$