Name_____ ID_____

MATH 311   Exam 1   Fall 2001
Section 200   Solutions   P. Yasskin

1	/15	8	/ 5
2-5	/12	9	/15
6	/ 8	10	/15
7	/15	11	/15

In problems 1 through 6, let $A = \begin{pmatrix} 1 & -1 & 1 & -2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & -3 \\ 2 & 2 & 0 & p \end{pmatrix}$ and $\vec{X} = \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}$.

1. (15 points) Compute $\det A$.

$$\det A = \begin{vmatrix} 1 & -1 & 1 & -2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & -3 \\ 2 & 2 & 0 & p \end{vmatrix} \underset{=}{R_4 - 2R_1} \begin{vmatrix} 1 & -1 & 1 & -2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 4 & -2 & p+4 \end{vmatrix} \underset{=}{\text{expand } C_1} \begin{vmatrix} 1 & -1 & 3 \\ 0 & 1 & -3 \\ 4 & -2 & p+4 \end{vmatrix}$$

$$\underset{=}{R_3 - 4R_1} \begin{vmatrix} 1 & -1 & 3 \\ 0 & 1 & -3 \\ 0 & 2 & p-8 \end{vmatrix} \underset{=}{\text{expand } C_1} \begin{vmatrix} 1 & -3 \\ 2 & p-8 \end{vmatrix} = (p-8) - (-6) = p - 2$$

2. (3 points) No Part Credit. Circle the answer.

For what value of p is $\det A = 0$?

a) 0   b) 1   c) 2   d) 3   e) 4

3. (3 points) No Part Credit. Circle the answer.

For what value of p is $\det A = 1$?

a) 0   b) 1   c) 2   d) 3   e) 4

In problems 1 through 6, let $A = \begin{pmatrix} 1 & -1 & 1 & -2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & -3 \\ 2 & 2 & 0 & p \end{pmatrix}$ and $\vec{X} = \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}$.

4. (3 points) No Part Credit. Circle the answer.

With $p = 4$, how many solutions are there to the equations $A\vec{X} = \vec{0}$?

a) No Solutions | b) A Unique Solution | c) ∞-many Solutions

$\det A = p - 2 = 2 \neq 0 \;\Rightarrow\;$ A is invertible $\;\Rightarrow\; \vec{X} = A^{-1}\vec{0} = \vec{0}$ uniquely

5. (3 points) No Part Credit. Circle the answer.

With $p = 2$, how many solutions are there to the equations $A\vec{X} = \vec{0}$?

a) No Solutions   b) A Unique Solution | c) ∞-many Solutions |

$\det A = 0 \;\Rightarrow\;$ A is non-invertible

$\Rightarrow\;$ No Solutions or ∞-many Solutions, but $\vec{X} = \vec{0}$ is a solution

6. (8 points) Solve **ONE** of the following two problems.
You should be able to just write down the solution.

a.   Find all soultions of the equations $A\vec{X} = \vec{0}$ with $p = 4$.

$\det A = p - 2 = 2 \neq 0 \;\Rightarrow\;$ A is invertible $\;\Rightarrow\; \vec{X} = A^{-1}\vec{0} = \vec{0}$ uniquely

OR

b.   Find all soultions of the equations $A\vec{X} = \vec{0}$ with $p = 2$.

$\left(\begin{array}{cccc|c} 1 & -1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 2 & 2 & 0 & 2 & 0 \end{array}\right) \underset{\Rightarrow}{\overset{R_4-2R_1}{}} \left(\begin{array}{cccc|c} 1 & -1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 4 & -2 & 6 & 0 \end{array}\right) \underset{\Rightarrow}{\overset{R_1+R_2}{\underset{}{R_4-4R_4}}} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 2 & -6 & 0 \end{array}\right)$

$\underset{\Rightarrow}{\overset{R_2+R_3}{\underset{}{R_4-2R_3}}} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \;\Rightarrow\; \begin{array}{l} w = -t \\ x = 0 \\ y = 3t \\ z = t \end{array}$

2

In problems 7 and 8, let $C = \begin{pmatrix} -1 & 1 & -1 \\ 3 & -2 & 3 \\ -6 & 4 & -5 \end{pmatrix}$, $\vec{B} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\vec{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

7. (15 points) Find C^{-1}.

$$\left(\begin{array}{ccc|ccc} -1 & 1 & -1 & 1 & 0 & 0 \\ 3 & -2 & 3 & 0 & 1 & 0 \\ -6 & 4 & -5 & 0 & 0 & 1 \end{array}\right) \overset{\substack{-R_1 \\ R_2+3R_1 \\ R_3-6R_1}}{\Longrightarrow} \left(\begin{array}{ccc|ccc} 1 & -1 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 3 & 1 & 0 \\ 0 & -2 & 1 & -6 & 0 & 1 \end{array}\right)$$

$$\overset{\substack{R_1+R_2 \\ R_3+2R_2}}{\Longrightarrow} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 3 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & 1 \end{array}\right) \overset{R_1-R_3}{\Longrightarrow} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & -1 \\ 0 & 1 & 0 & 3 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & 1 \end{array}\right)$$

$$C^{-1} = \begin{pmatrix} 2 & -1 & -1 \\ 3 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$$

Check:

$$C^{-1}C = \begin{pmatrix} 2 & -1 & -1 \\ 3 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 & -1 \\ 3 & -2 & 3 \\ -6 & 4 & -5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

8. (5 points) Solve $C\vec{X} = \vec{B}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \vec{X} = C^{-1}\vec{B} = \begin{pmatrix} 2 & -1 & -1 \\ 3 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ 5 \\ 7 \end{pmatrix}$$

3

9. (15 points) Find the point(s), if any, where the line $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$

intersects the plane $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + r \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + s \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Equate x, y and z. Solve for r, s and t. Substitute back to get x, y and z.

$$\left\{ \begin{array}{l} 1 + s = 3 + 2t \\ 2 + r + s = -2 + t \\ 3 + r = 1 - t \end{array} \right. \Rightarrow \left\{ \begin{array}{l} s - 2t = 2 \\ r + s - t = -4 \\ r + t = -2 \end{array} \right.$$

$$\left(\begin{array}{ccc|c} 0 & 1 & -2 & 2 \\ 1 & 1 & -1 & -4 \\ 1 & 0 & 1 & -2 \end{array} \right) \begin{array}{c} R_3 \\ R_1 \\ R_2 \end{array} \Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & -2 \\ 0 & 1 & -2 & 2 \\ 1 & 1 & -1 & -4 \end{array} \right) \overset{R_3 - R_1 - R_2}{\Rightarrow} \left(\begin{array}{ccc|c} 1 & 0 & 1 & -2 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & -4 \end{array} \right)$$

No solutions! The line does not intersect the plane.

10. (15 points) A compact disk is 5 cm in radius. As its rotation speeds up, a speck of dust on the edge moves along the curve

$$\vec{r}(t) = \left(5\cos(t^2), 5\sin(t^2) \right)$$

where t is in sec. At $t = 2$ sec, the speck of dust flies off the disk and travels along the tangent line with constant velocity (equal to its velocity at the time of release). Where is the dust particle at $t = 3$ sec?

$\vec{r}(t) = \left(5\cos(t^2), 5\sin(t^2) \right)$ 　　　　$\vec{r}(2) = \left(5\cos 4, \ 5\sin 4 \right)$

$\vec{v}(t) = \left(-10t\sin(t^2), 10t\cos(t^2) \right)$ 　　　$\vec{v}(2) = \left(-20\sin 4, \ 20\cos 4 \right)$

$\vec{r}_{\tan}(t) = \vec{r}(2) + \vec{v}(2) \cdot (t - 2)$

$\vec{r}_{\tan}(3) = \vec{r}(2) + \vec{v}(2) \cdot (3 - 2) = \left(5\cos 4, \ 5\sin 4 \right) + \left(-20\sin 4, \ 20\cos 4 \right)$

$(x, y) = \left(5\cos 4 - 20\sin 4, \ 5\sin 4 + 20\cos 4 \right)$

4

11. (15 points) An ant is sitting on a frying pan at the point $(x,y) = (2,1)$. At this point the temperature is $T(2,1) = 105°F$ and the gradient of the temperature is $\nabla T(2,1) = \left(-2\frac{°F}{cm}, -3\frac{°F}{cm}\right)$.

    a. (3 pts) In what direction should the ant walk to decrease the temperature as fast as possible?

    Move in the direction opposite to the gradient: $-\nabla T(2,1) = (2,3)$

    b. (4 pts) If the ant's velocity is $\vec{v} = \left(\frac{dx}{dt}, \frac{dy}{dt}\right) = \left(-.4\frac{cm}{sec}, .6\frac{cm}{sec}\right)$, what is the rate of change of the temperature, $\frac{dT}{dt}$, at the ant's location?

$$\frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} = \nabla T \cdot \vec{v} = (-2,-3) \cdot (-.4,.6) = .8 - 1.8 = -1\frac{°F}{sec}$$

    c. (4 pts) Write out the linear approximation to the temperature near the point $(x,y) = (2,1)$. NOTE: This is the same as the equation of the tangent plane to the graph of $z = T(x,y)$ at the point $(x,y) = (2,1)$.

$$T_{lin}(x,y) = T(2,1) + \nabla T(2,1)\begin{pmatrix} x-2 \\ y-1 \end{pmatrix} = 105 + (-2,-3)\begin{pmatrix} x-2 \\ y-1 \end{pmatrix}$$
$$= 105 - 2(x-2) - 3(y-1)$$

    d. (4 pts) If the ant walks to the point $(x,y) = (1.8, 1.3)$, by about how much will the temperature increase or decrease?

$$\Delta T(x,y) = T_{lin}(x,y) - T(2,1) = \nabla T(2,1)\begin{pmatrix} x-2 \\ y-1 \end{pmatrix} = -2(x-2) - 3(y-1)$$
$$\Delta T(1.8, 1.3) = -2(1.8-2) - 3(1.3-1) = -2(-.2) - 3(.3) = .4 - .9 = -.5$$
The temperature will decrease by $.5°F$.