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| 1 | /10 | 4 | /10 |
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| 3 | /10 | | |

MATH 311 Exam 2 Fall 2001
Section 200 Solutions P. Yasskin

**1.** (10 points) Let $P_1$ be the vector space of polynomials of degree $\le 1$. Suppose $L : P_1 \to \mathbf{R}$ is a linear map which satisfies

$$L(2+3t) = 1, \qquad L(1+4t) = -2.$$

Compute $L(5-2t)$.

$$5-2t = a(2+3t) + b(1+4t) = (2a+b) + (3a+4b)t$$

$$\begin{aligned} 2a+b &= 5 \\ 3a+4b &= -2 \end{aligned} \qquad \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}^{-1}\begin{pmatrix} 5 \\ -2 \end{pmatrix} = \frac{1}{8-3}\begin{pmatrix} 4 & -1 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 5 \\ -2 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 22 \\ -19 \end{pmatrix}$$

$$5-2t = \frac{22}{5}(2+3t) - \frac{19}{5}(1+4t)$$

$$L(5-2t) = \frac{22}{5}L(2+3t) - \frac{19}{5}L(1+4t) = \frac{22}{5}1 - \frac{19}{5}(-2) = \frac{22+38}{5} = 12$$

**2.** (10 points) Which of the following is not a subspace of $C^1[-1,1]$ ? Why?

$$P = \{f \in C^1[-1,1] \mid f(-1) = f(1)\} \qquad Q = \left\{f \in C^1[-1,1] \mid \frac{f(-1)+f(1)}{2} = f(0)\right\}$$

$$R = \left\{f \in C^1[-1,1] \mid \int_0^1 f(t)\,dt = 1\right\} \qquad S = \{f \in C^1[-1,1] \mid f'(0) = f(0)\}$$

$R$ is not a subspace because if $f,g \in R$, then $\int_0^1 f(t)\,dt = 1$ and $\int_0^1 g(t)\,dt = 1$
but $\int_0^1 (f+g)(t)\,dt = 2$. So $f+g \notin R$.

**3.** $(10 \text{ points})$ Duke Skywater is flying the Millennium Eagle through the Asteroid Belt. At the current time, his position is $\vec{r} = (4,-1,2)$ and his velocity is $\vec{v} = (3,2,-1)$. He measures that the electric field and its Jacobian are currently

$$\vec{E} = \begin{pmatrix} 12 \\ 2 \\ 9 \end{pmatrix} \quad \text{and} \quad J\vec{E} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 4 & 3 \\ 1 & 2 & 9 \end{pmatrix}.$$

Use a linear (affine) approximation to estimate what the electric field will be $2$ sec from now.

$$\vec{E}(\vec{r}(t)) \approx \vec{E}(\vec{r}(t_0)) + J\vec{E} \cdot (\vec{r}(t) - \vec{r}(t_0)) \approx \vec{E}(\vec{r}(t_0)) + J\vec{E} \cdot \vec{v}(t_0)(t - t_0)$$

$$\approx \begin{pmatrix} 12 \\ 2 \\ 9 \end{pmatrix} + \begin{pmatrix} 2 & 0 & 1 \\ 0 & 4 & 3 \\ 1 & 2 & 9 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} (2) = \begin{pmatrix} 12 \\ 2 \\ 9 \end{pmatrix} + 2 \begin{pmatrix} 5 \\ 5 \\ -2 \end{pmatrix} = \begin{pmatrix} 22 \\ 12 \\ 5 \end{pmatrix}$$

**4.** $(10 \text{ points})$ Let $L : R^5 \to R^4$ be a linear map whose matrix is $A$. If $A$ is row reduced, one obtains the matrix

$$\begin{pmatrix} 1 & 3 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

What is the dimension of the kernel of $L$? What is the dimension of the image of $L$? Be sure to explain why.

$Ker(L) = \{X \mid AX = 0\}$ This has $2$ free parameters since there are $2$ columns without leading $1$'s. So $\dim Ker(L) = 2$.

$\dim Im(L) =$ # linearly independent columns $=$ column rank $=$ row rank
$=$ # linearly independent rows $=$ # leading $1$'s $= 3$.

It is OK to do one of these and then use the Nullity-Rank Theorem:
$\dim Ker(L) + \dim Im(L) = \dim Dom(L) = 5$.

**5.** (60 points) Let $M(2,2)$ be the vector space of $2 \times 2$ matrices. Let $P_2$ be the vector space of polynomials of degree $\le 2$. Consider the linear map $L : M(2,2) \to P_2$ given by

$$L(M) = \begin{pmatrix} 1 & x \end{pmatrix} M \begin{pmatrix} 1 \\ x \end{pmatrix}$$

Hint: For some parts it may be useful to write $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and/or $p(x) = \alpha + \beta x + \gamma x^2$.

**a.** (3) Identify the domain of $L$, a basis for the domain, and the dimension of the domain.

$$Dom(L) = M(2,2) \qquad Basis\_Dom(L) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$$\dim Dom(L) = 4$$

**b.** (3) Identify the codomain of $L$, a basis for the codomain, and the dimension of the codomain.

$$Codom(L) = P_2 \qquad Basis\_Codom(L) = \{1, x, x^2\} \qquad \dim Codom(L) = 3$$

**c.** (6) Identify the kernel of $L$, a basis for the kernel, and the dimension of the kernel.

$$L\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} 1 & x \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ x \end{pmatrix} = \begin{pmatrix} 1 & x \end{pmatrix}\begin{pmatrix} a + bx \\ c + dx \end{pmatrix} = a + (b+c)x + dx^2$$

$$Ker(L) = \{M \mid L(M) = 0\} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a + (b+c)x + dx^2 = 0 \right\} = \left\{ \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \right\}$$

$$= Span\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\} \qquad Basis\_Ker(L) = \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\} \qquad \dim Ker(L) = 1$$

**d.** (6) Identify the image of $L$, a basis for the image, and the dimension of the image.

$$Im(L) = \{L(M)\} = \{a + (b+c)x + dx^2\} = Span\{1, x, x^2\} = P_2$$

$$Basis\_Im(L) = \{1, x, x^2\} \qquad \dim Im(L) = 3$$

**e.** (2) Is the function $L$ one-to-one? Why?

$L$ is not one-to-one because $Ker(L) \neq \{0\}$.

**f.** (2) Is the function $L$ onto? Why?

$L$ is onto because $Im(L) = P_2 = Codom(L)$.

**g.** (2) Verify the dimensions in a, b, c and d agree with the Nullity-Rank Theorem.

$$\dim Ker(L) + \dim Im(L) = \dim Dom(L) \qquad 1 + 3 = 4$$

**h.** (6) Find the matrix of $L$ relative to the standard bases: (Call it $\underset{E\leftarrow e}{A}$.)

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad e_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{for} \quad M(2,2)$$

$$\text{and} \quad E_1 = 1, \quad E_2 = x, \quad E_3 = x^2 \quad \text{for} \quad P_2$$

$$L\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} 1 & x \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix} = \begin{pmatrix} 1 & x \end{pmatrix} \begin{pmatrix} a+bx \\ c+dx \end{pmatrix} = a + (b+c)x + dx^2$$

$$L(e_1) = 1 = E_1$$
$$L(e_2) = x = E_2$$
$$L(e_3) = x = E_2$$
$$L(e4) = x^2 = E_3$$

$$\underset{E\leftarrow e}{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

**i.** (6) Another basis for $P_2$ is $F_1 = 1 + x$, $F_2 = 1 + x^2$, $F_3 = x + x^2$. Find the change of basis matrices between the $E$ and $F$ bases. (Call them $\underset{F\leftarrow E}{C}$ and $\underset{E\leftarrow F}{C}$.) Be sure to identify which is which!

$$F_1 = 1 + x = E_1 + E_2$$
$$F_2 = 1 + x^2 = E_1 + E_3$$
$$F_3 = x + x^2 = E_2 + E_3$$

$$\underset{E\leftarrow F}{C} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\substack{R_2 \\ R_3 \\ R_1}} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \end{array}\right) \xrightarrow{R_3 - R_1 - R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & -2 & 1 & -1 & -1 \end{array}\right)$$

$$\xrightarrow{\frac{-1}{2}R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{-1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right) \xrightarrow{\substack{R_1 - R_3 \\ R_2 - R_3}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{-1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right)$$

$$\underset{F\leftarrow E}{C} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

**j.** (6) Consider the polynomial $q = 2 + 4x$. Find $[q]_E$ and $[q]_F$, the components of $q$ relative to the $E$ and $F$ bases, respectively. Check $[q]_F$.

$$q = 2E_1 + 4E_2 \qquad [q]_E = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} \qquad [q]_F = \underset{F\leftarrow E}{C}\,[q]_E = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$$

$$3F_1 - 1F_2 + 1F_3 = 3(1+x) - 1(1+x^2) + 1(x+x^2) = 2 + 4x = q$$

**k.** $(5)$ Find the matrix of $L$ relative to the $e$ basis for $M(2,2)$ and the $F$ basis for $P_2$. (Call it $\underset{F\leftarrow e}{B}$.)

$$\underset{F\leftarrow e}{B} = \underset{F\leftarrow E}{C}\ \underset{E\leftarrow e}{A} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{-1}{2} & \frac{-1}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

**l.** $(5)$ Find $\underset{F\leftarrow e}{B}$ by a second method.

$$\begin{aligned}
L(e_1) = 1 &= aF_1 + bF_2 + cF_3 = a(1+x) + b(1+x^2) + c(x+x^2) = \tfrac{1}{2}(1+x) + \tfrac{1}{2}(1+x^2) - \tfrac{1}{2}(x+x^2) \\
L(e_2) = x &= dF_1 + eF_2 + fF_3 = d(1+x) + e(1+x^2) + f(x+x^2) = \tfrac{1}{2}(1+x) - \tfrac{1}{2}(1+x^2) + \tfrac{1}{2}(x+x^2) \\
L(e_3) = x &= dF_1 + eF_2 + fF_3 = d(1+x) + e(1+x^2) + f(x+x^2) = \tfrac{1}{2}(1+x) - \tfrac{1}{2}(1+x^2) + \tfrac{1}{2}(x+x^2) \\
L(e4) = x^2 &= gF_1 + hF_2 + iF_3 = g(1+x) + h(1+x^2) + i(x+x^2) = -\tfrac{1}{2}(1+x) + \tfrac{1}{2}(1+x^2) + \tfrac{1}{2}(x+x^2)
\end{aligned}$$

$$\underset{F\leftarrow e}{B} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{-1}{2} & \frac{-1}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

**m.** $(6)$ Consider the matrix $N = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Find $[N]_e$, the components of $N$ relative to the $e$-basis and $[L(N)]_F$, the components of $L(N)$ relative to the $F$ basis. Use $[L(N)]_F$ to find $L(N)$?

$$N = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = 1e_1 + 2e_2 + 3e_3 + 4e_4 \qquad [N]_e = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

$$[L(N)]_F = \underset{F\leftarrow e}{B}\ [N]_e = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{-1}{2} & \frac{-1}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}$$

$$L(N) = 1F_1 + 4F_3 = 1(1+x) + 4(x+x^2) = 1 + 5x + 4x^2$$

**n.** $(2)$ Recompute $L(N)$ using the definition of $L$.

$$L(N) = \begin{pmatrix} 1 & x \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix} = 1 + 5x + 4x^2$$