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MATH 311 Exam 3 Fall 2001
Section 200 Solutions P. Yasskin

1. (35 points) Compute $\iint_{\partial C} \vec{F} \cdot d\vec{S}$ over the complete surface of the box

$$0 \le x \le 2 \qquad 0 \le y \le 3 \qquad 0 \le z \le 4$$

where $\vec{F} = (x^2y^2z^3, xy^3z^3, xy^2z^4)$.

$$\nabla \cdot \vec{F} = \nabla \cdot (x^2y^2z^3, xy^3z^3, xy^2z^4) = 2xy^2z^3 + 3xy^2z^3 + 4xy^2z^3 = 9xy^2z^3$$

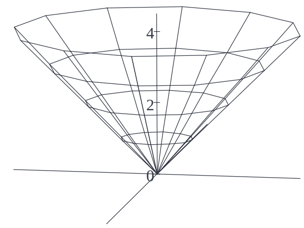
By Gauss' Theorem

$$\iint_{\partial C} \vec{F} \cdot d\vec{S} = \iiint_C \vec{\nabla} \cdot F\, dV = \int_0^4 \int_0^3 \int_0^2 9xy^2z^3\, dx\, dy\, dz = 9\left[\frac{x^2}{2}\right]_0^2 \left[\frac{y^3}{3}\right]_0^3 \left[\frac{z^4}{4}\right]_0^4$$

$$= 9 \cdot 2 \cdot 9 \cdot 64 = 10368$$

2. (35 points) Consider the cone C given by

$$z = \sqrt{x^2 + y^2} \quad \text{for } z \le 4$$



and the vector field $\vec{F} = (-yz, xz, -xy)$.

We want to compute $\iint\limits_C \vec{\nabla} \times \vec{F} \cdot d\vec{S}$ with normal pointing up and into the cone.

a. (5) Compute $\vec{\nabla} \times \vec{F}$.

$$\vec{\nabla} \times (-yz, xz, xy) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ -yz & xz & -xy \end{vmatrix} = (-2x, 0, 2z)$$

b. (10) Parametrize the cone using cylindrical coordinates r and θ as the parameters and give the range of the parameters. Then explicitly compute $\iint\limits_C \vec{\nabla} \times \vec{F} \cdot d\vec{S}$.

$\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, r)$ with $0 \le r \le 4$, $0 \le \theta \le 2\pi$

$$\begin{array}{lccc} & \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \vec{R}_r = (& \cos\theta, & \sin\theta, & 1 \ ) \\ \vec{R}_\theta = (& -r\sin\theta, & r\cos\theta, & 0 \ ) \end{array}$$

$\vec{N} = \vec{R}_r \times \vec{R}_\theta = (-r\cos\theta, -r\sin\theta, r)$ which points up and in.

$$\vec{\nabla} \times \vec{F} = (-2x, 0, 2z) = (-2r\cos\theta, 0, 2r)$$

$$\iint\limits_C \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \iint\limits_C \vec{\nabla} \times \vec{F} \cdot \vec{N}\, dr\, d\theta = \int_0^{2\pi} \int_0^4 (2r^2\cos^2\theta + 2r^2)\, dr\, d\theta$$

$$= 2\frac{r^3}{3}\bigg|_0^4 \int_0^{2\pi} (\cos^2\theta + 1)\, d\theta = \frac{128}{3}(\pi + 2\pi) = 128\pi$$

RECALL: C is the cone $z = \sqrt{x^2 + y^2}$ for $z \le 4$ with normal pointing up and into the cone and $\vec{F} = (-yz, xz, -xy)$.

c. (10) Describe 2 other ways to compute $\iint_C \vec{\nabla} \times \vec{F} \cdot d\vec{S}$. Be sure to name or quote any Theorem you use and discuss the orientation of any curves or surfaces.

i. By Stokes' Theorem, $\iint_C \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \int_{\partial C} \vec{F} \cdot d\vec{s}$ where ∂C is the circle $x^2 + y^2 = 16$ and $z = 4$ traversed counterclockwise.

ii. By Stokes' Theorem again, $\int_{\partial C} \vec{F} \cdot d\vec{s} = \iint_D \vec{\nabla} \times \vec{F} \cdot d\vec{S}$ where D is the disk $x^2 + y^2 \le 16$ and $z = 4$ with normal pointing up.

d. (10) Recompute $\iint_C \vec{\nabla} \times \vec{F} \cdot d\vec{S}$ by **one** of these two methods.

i. $\vec{r}(t) = (4\cos\theta, 4\sin\theta, 4)$

$\vec{v}(t) = (-4\sin\theta, 4\cos\theta, 0)$

$\vec{F} = (-yz, xz, -xy) = (-16\sin\theta, 16\cos\theta, -16\sin\theta\cos\theta)$

$$\int_{\partial C} \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F} \cdot \vec{v}\, d\theta = \int_0^{2\pi} \left(64\sin^2\theta + 64\cos^2\theta\right) d\theta = 128\pi$$

OR

ii. We parametrize the disk D.

$\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, 4)$ with $0 \le r \le 4$, $0 \le \theta \le 2\pi$

$$\begin{array}{rcccc} & \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} & \\ \vec{R}_r = (& \cos\theta, & \sin\theta, & 0 &) \\ \vec{R}_\theta = (& -r\sin\theta, & r\cos\theta, & 0 &) \end{array}$$

$\vec{N} = \vec{R}_r \times \vec{R}_\theta = (0, 0, r)$ which points up.

$\vec{\nabla} \times \vec{F} = (-2x, 0, 2z) = (-2r\cos\theta, 0, 8)$

$$\iint_D \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \iint_D \vec{\nabla} \times \vec{F} \cdot \vec{N}\, dr\, d\theta = \int_0^{2\pi} \int_0^4 (8r)\, dr\, d\theta = 2\pi \left[4r^2\right]_0^4 = 128\pi$$

3. $\left(30 \text{ points}\right)$ A hypersurface S in $\mathbf{R}^4$ with coordinates (w,x,y,z), may be parametrized by

$$(w,x,y,z) = \vec{R}(r,\theta,\varphi) = (r\cos\theta, r\sin\theta, r\cos\varphi, r\sin\varphi)$$

for $0 \le r \le 3,\quad 0 \le \theta \le 2\pi$ and $0 \le \varphi \le 2\pi$.

a. (15) Find the tangent vectors, the normal vector and length of the normal vector.

$$\begin{array}{rcccccc} & & \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} & \hat{\mathbf{l}} & \\ \vec{R}_r & = (& \cos\theta, & \sin\theta, & \cos\varphi, & \sin\varphi &) \\ \vec{R}_\theta & = (& -r\sin\theta, & r\cos\theta, & 0, & 0 &) \\ \vec{R}_\varphi & = (& 0, & 0, & -r\sin\varphi, & r\cos\varphi &) \end{array}$$

$$\vec{N} = \hat{\mathbf{i}}\begin{vmatrix} \sin\theta & \cos\varphi & \sin\varphi \\ r\cos\theta & 0 & 0 \\ 0, & -r\sin\varphi & r\cos\varphi \end{vmatrix} - \hat{\mathbf{j}}\begin{vmatrix} \cos\theta & \cos\varphi & \sin\varphi \\ -r\sin\theta & 0 & 0 \\ 0 & -r\sin\varphi & r\cos\varphi \end{vmatrix}$$

$$+\hat{\mathbf{k}}\begin{vmatrix} \cos\theta & \sin\theta & \sin\varphi \\ -r\sin\theta & r\cos\theta & 0 \\ 0 & 0 & r\cos\varphi \end{vmatrix} - \hat{\mathbf{l}}\begin{vmatrix} \cos\theta & \sin\theta & \cos\varphi \\ -r\sin\theta & r\cos\theta & 0 \\ 0 & 0 & -r\sin\varphi \end{vmatrix}$$

$$= \hat{\mathbf{i}}\left[-r\cos\theta\left(r\cos^2\varphi + r\sin^2\varphi\right)\right] - \hat{\mathbf{j}}\left[r\sin\theta\left(r\cos^2\varphi + r\sin^2\varphi\right)\right]$$
$$+\hat{\mathbf{k}}\left[r\cos\varphi\left(r\cos^2\theta + r\sin^2\theta\right)\right] - \hat{\mathbf{l}}\left[-r\sin\varphi\left(r\cos^2\theta + r\sin^2\theta\right)\right]$$

$$= (-r^2\cos\theta, -r^2\sin\theta, r^2\cos\varphi, r^2\sin\varphi)$$

$$\left|\vec{N}\right| = \sqrt{(-r^2\cos\theta)^2 + (-r^2\sin\theta)^2 + (r^2\cos\varphi)^2 + (r^2\sin\varphi)^2} = \sqrt{2r^4} = r^2\sqrt{2}$$

b. (5) Find the hyperarea of the hypersurface.

$$A = \iiint_S \left|\vec{N}\right| dr\,d\theta\,d\varphi = \int_0^{2\pi}\int_0^{2\pi}\int_0^3 r^2\sqrt{2}\,dr\,d\theta\,d\varphi = (2\pi)^2\sqrt{2}\,\frac{r^3}{3}\Big|_0^3 = 36\pi^2\sqrt{2}$$

RECALL: S is the hypersurface parametrized by

$$(w,x,y,z) = \vec{R}(r,\theta,\varphi) = (r\cos\theta, r\sin\theta, r\cos\varphi, r\sin\varphi)$$

for $0 \le r \le 3$, $0 \le \theta \le 2\pi$ and $0 \le \varphi \le 2\pi$.

c. (5) Compute $P = \iiint_S \sqrt{2w^2 + 2x^2}\, dS$ over the hypersurface.

$$P = \iiint_S \sqrt{2r^2}\, \left|\vec{N}\right| dr\,d\theta\,d\varphi = \int_0^{2\pi}\int_0^{2\pi}\int_0^3 2r^3 dr\,d\theta\,d\varphi = (2\pi)^2 2\frac{r^4}{4}\Big|_0^3 = 162\pi^2$$

d. (5) Compute $Q = \iiint_S (w\,dy\,dx\,dz - 5z\,dw\,dx\,dy)$ over the hypersurface.

$$Q = \iiint_S \left[(r\cos\theta)\frac{\partial(y,x,z)}{\partial(r,\theta,\varphi)} - 5(r\sin\varphi)\frac{\partial(w,x,y)}{\partial(r,\theta,\varphi)} \right] dr\,d\theta\,d\varphi$$

$$\frac{\partial(y,x,z)}{\partial(r,\theta,\varphi)} = -\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} = -N_1 = r^2\cos\theta \qquad \frac{\partial(w,x,y)}{\partial(r,\theta,\varphi)} = -N_4 = -r^2\sin\varphi$$

$$Q = \iiint_S [(r\cos\theta)(r^2\cos\theta) - 5(r\sin\varphi)(-r^2\sin\varphi)]\,dr\,d\theta\,d\varphi$$

$$= \int_0^{2\pi}\int_0^{2\pi}\int_0^3 \left[r^3\cos^2\theta + 5r^3\sin^2\varphi\right] dr\,d\theta\,d\varphi = 2\pi\frac{r^4}{4}\Big|_0^3 \left[\int_0^{2\pi}\cos^2\theta\,d\theta + 5\int_0^{2\pi}\sin^2\varphi\,d\varphi\right]$$

$$= \pi\frac{81}{2}(\pi + 5\pi) = 243\pi^2$$