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1	/10	4	/20
2	/20	5-7	/30
3	/20		

MATH 311 Final Exam Fall 2001
Section 200 Solutions P. Yasskin

1. $\left(10\right.$ points$\left.\right)$ Let P_2^0 be the subset of P_2 consisting of those polynomials of degree 2 or less whose constant term is zero. In particular

$$P_2^0 = \{p = ax + bx^2\}$$

a. (8) Show P_2^0 is a subspace of P_2.

Let $p,q \in P_2^0$. Then $p = ax + bx^2$ and $q = cx + dx^2$. So $p + q = (a+c)x + (b+d)x^2 \in P_2^0$. Further $kp = (ka)x + (kb)x^2 \in P_2^0$. So P_2^0 is closed under addition and scalar multiplication and so is a subspace.

b. (2) What is the dimension of P_2^0? Why?

$\dim P_2^0 = 2$ because a basis is $\{x, x^2\}$ which has 2 vectors.

2. (20 points) Again let P_2^0 be the subset of P_2 consisting of those polynomials of degree 2 or less whose constant term is zero. Consider the function $\langle *, * \rangle$ of two polynomials given by

$$\langle p, q \rangle = \int_0^1 \frac{4pq}{x^2}\, dx.$$

a. (5) Show the function $\langle *, * \rangle$ is an inner product on P_2^0.

i. symmetric: $\langle q, p \rangle = \int_0^1 \frac{4qp}{x^2}\, dx = \int_0^1 \frac{4pq}{x^2}\, dx = \langle p, q \rangle$

ii. bilinear: $\langle p, aq + br \rangle = \int_0^1 \frac{4p(aq + br)}{x^2}\, dx = a\int_0^1 \frac{4pq}{x^2}\, dx + b\int_0^1 \frac{4pr}{x^2}\, dx = a\langle p, q \rangle + b\langle p, r \rangle$

iii. positive definite: $\langle p, p \rangle = \int_0^1 \frac{4p^2}{x^2}\, dx \geq 0$ because the integral of a non-negative quantity is non-negative.

If $\langle p, p \rangle = \int_0^1 \frac{4p^2}{x^2}\, dx = 0$, then $\frac{4p^2}{x^2} = 0$, or $p = 0$.

b. (10) Apply the Gram Schmidt procedure to the basis $p_1 = x,\quad p_2 = x^2$ to produce an orthogonal basis q_1, q_2 and an orthonormal basis r_1, r_2.

$$q_1 = p_1 = x$$
$$\langle q_1, q_1 \rangle = \int_0^1 \frac{4q_1^2}{x^2}\, dx = \int_0^1 \frac{4x^2}{x^2}\, dx = \int_0^1 4\, dx = 4x|_0^1 = 4$$
$$|q_1| = 2$$
$$r_1 = \frac{q_1}{|q_1|} = \frac{x}{2}$$
$$\langle p_2, q_1 \rangle = \int_0^1 \frac{4p_2 q_1}{x^2}\, dx = \int_0^1 \frac{4x^2 x}{x^2}\, dx = \int_0^1 4x\, dx = 2x^2|_0^1 = 2$$
$$q_2 = p_2 - \frac{\langle p_2, q_1 \rangle}{\langle q_1, q_1 \rangle} q_1 = x^2 - \frac{2}{4}x = x^2 - \frac{x}{2}$$
$$\langle q_2, q_2 \rangle = \int_0^1 \frac{4q_2^2}{x^2}\, dx = \int_0^1 \frac{4\left(x^2 - \frac{x}{2}\right)^2}{x^2}\, dx = \int_0^1 4\left(x - \frac{1}{2}\right)^2 dx = \left.\frac{4\left(x - \frac{1}{2}\right)^3}{3}\right|_0^1 = \frac{1}{6} - \frac{-1}{6} = \frac{1}{3}$$
$$|q_2| = \frac{1}{\sqrt{3}}$$
$$r_2 = \frac{q_2}{|q_2|} = \sqrt{3}\left(x^2 - \frac{x}{2}\right)$$

Summary:

$$q_1 = x, \qquad q_2 = x^2 - \frac{x}{2} \qquad r_1 = \frac{x}{2}, \qquad r_2 = \sqrt{3}\left(x^2 - \frac{x}{2}\right)$$

c. (5) Find the change of basis matrices $\underset{r \leftarrow p}{C}$ and $\underset{p \leftarrow r}{C}$.

$$r_1 = \frac{x}{2} = \frac{1}{2}p_1 + 0p_2$$
$$r_2 = \sqrt{3}\left(x^2 - \frac{x}{2}\right) = -\frac{\sqrt{3}}{2}p_1 + \sqrt{3}p_2$$

$$\underset{p \leftarrow r}{C} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \sqrt{3} \end{pmatrix}$$

$$\underset{r \leftarrow p}{C} = \underset{p \leftarrow r}{C}^{-1} = \frac{1}{\frac{\sqrt{3}}{2}} \begin{pmatrix} \sqrt{3} & \frac{\sqrt{3}}{2} \\ 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & \frac{1}{\sqrt{3}} \end{pmatrix}$$

3. (20 points) Again let P_2^0 be the subset of P_2 consisting of those polynomials of degree 2 or less whose constant term is zero. Consider the function $L : P_2^0 \to P_2$ given by

$$L(p) = p - \frac{dp}{dx}.$$

a. (4) Show the function L is linear.

$$L(ap + bq) = (ap + bq) - \frac{d(ap + bq)}{dx} = a\left(p - \frac{dp}{dx}\right) + b\left(q - \frac{dq}{dx}\right) = aL(p) + bL(q)$$

b. (6) Find the kernel of L. Give a basis.

Let $p = ax + bx^2$. Then $L(p) = 0$ says

$$(ax + bx^2) - \frac{d(ax + bx^2)}{dx} = (ax + bx^2) - (a + 2bx) = (-a) + (a - 2b)x + bx^2 = 0.$$

This implies $a = b = 0$, and so $p = 0$.

Therefore $Ker(L) = \{0\}$. There is no basis.

c. (6) Find the image of L. Give a basis.

$$L(p) = (ax + bx^2) - \frac{d(ax + bx^2)}{dx} = (ax + bx^2) - (a + 2bx) = a(x - 1) + b(x^2 - 2x)$$

Therefore $Im(L) = \{a(x - 1) + b(x^2 - 2x)\} = Span(x - 1, x^2 - 2x)$

Basis is $\{x - 1, x^2 - 2x\}$.

d. (2) Is L onto? Why?

L is not onto because $Codom(L) = P_2$ while $Im(L) = Span(x - 1, x^2 - 2x) \neq P_2$

e. (2) Is L one-to-one? Why?

L is one-to-one because $Ker(L) = \{0\}$.

4. (20 points) Again let P_2^0 be the subset of P_2 consisting of those polynomials of degree 2 or less whose constant term is zero. Again consider the function $L : P_2^0 \to P_2$ given by

$$L(p) = p - \frac{dp}{dx}.$$

a. (10) Find the matrix of L relative to the bases

$$p_1 = x, \quad p_2 = x^2 \quad \text{for } P_2^0 \qquad \text{and} \qquad e_1 = 1, \quad e_2 = x, \quad e_3 = x^2 \quad \text{for } P_2.$$

Call it A.

$$L(p_1) = L(x) = x - \frac{dx}{dx} = x - 1 = -e_1 + e_2$$
$$L(p_2) = L(x^2) = x^2 - \frac{dx^2}{dx} = x^2 - 2x = -2e_2 + e_3$$

$$\underset{e\leftarrow p}{A} = \begin{pmatrix} -1 & 0 \\ 1 & -2 \\ 0 & 1 \end{pmatrix}$$

b. (5) Find the matrix of L relative to the bases

$$r_1, r_2 \quad \text{for } P_2^0 \qquad \text{and} \qquad e_1 = 1, \quad e_2 = x, \quad e_3 = x^2 \quad \text{for } P_2$$

where r_1, r_2 is the orthonormal basis you found in problem 2. Call it B.

$$\underset{e\leftarrow r}{B} = \underset{e\leftarrow p}{A} \; \underset{p\leftarrow r}{C} = \begin{pmatrix} -1 & 0 \\ 1 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \sqrt{3} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{1}{2} & -\frac{5\sqrt{3}}{2} \\ 0 & \sqrt{3} \end{pmatrix}$$

c. (5) Recompute B by another method.

$$L(r_1) = L\left(\frac{x}{2}\right) = \frac{x}{2} - \frac{d\frac{x}{2}}{dx} = \frac{x}{2} - \frac{1}{2} = -\frac{1}{2}e_1 + \frac{1}{2}e_2$$

$$L(r_2) = L\left(\sqrt{3}\left(x^2 - \frac{x}{2}\right)\right) = \sqrt{3}\left(x^2 - \frac{x}{2}\right) - \frac{d\sqrt{3}\left(x^2 - \frac{x}{2}\right)}{dx}$$
$$= \sqrt{3}\left(x^2 - \frac{x}{2}\right) - \sqrt{3}\left(2x - \frac{1}{2}\right) = \frac{\sqrt{3}}{2}e_1 - \frac{5\sqrt{3}}{2}e_2 + \sqrt{3}\,e_3$$

$$\underset{e\leftarrow r}{B} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{1}{2} & -\frac{5\sqrt{3}}{2} \\ 0 & \sqrt{3} \end{pmatrix}$$

5. (30 points) Do this problem, if you did the Volume of Desserts or Planet X Project.

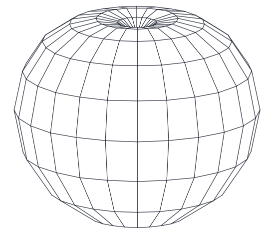
Find the z-component of the center of mass of the apple whose surface is given in spherical coordinates by

$$\rho = 1 - \cos\varphi$$

and whose density is 1.

HINT: The φ-integrals can be done using the substitution

$$u = 1 - \cos\varphi.$$



$$x = \rho\sin\varphi\cos\theta \qquad y = \rho\sin\varphi\sin\theta \qquad z = \rho\cos\varphi$$

$$dV = \rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$$

$$u = 1 - \cos\varphi \qquad du = \sin\varphi\, d\varphi$$

$$M = \iiint 1\, dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{1-\cos\varphi} \rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta = 2\pi\int_0^{\pi}\left[\frac{\rho^3}{3}\right]_0^{1-\cos\varphi}\sin\varphi\, d\varphi$$

$$= \frac{2\pi}{3}\int_0^{\pi}(1-\cos\varphi)^3\sin\varphi\, d\varphi = \frac{2\pi}{3}\frac{(1-\cos\varphi)^4}{4}\Bigg|_0^{\pi} = \frac{2\pi}{3}\frac{(2)^4}{4} = \frac{8\pi}{3}$$

$$z-mom = \iiint z\, dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{1-\cos\varphi} \rho\cos\varphi\,\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta = 2\pi\int_0^{\pi}\left[\frac{\rho^4}{4}\right]_0^{1-\cos\varphi}\cos\varphi\sin\varphi\, d\varphi$$

$$= \frac{\pi}{2}\int_0^{\pi}(1-\cos\varphi)^4\cos\varphi\sin\varphi\, d\varphi = \frac{\pi}{2}\int_0^{2}u^4(1-u)\, du = \frac{\pi}{2}\left[\frac{u^5}{5} - \frac{u^6}{6}\right]_0^2 = \frac{\pi}{2}\left[\frac{2^5}{5} - \frac{2^6}{6}\right]$$

$$= \frac{2^5\pi}{2}\left[\frac{1}{5} - \frac{1}{3}\right] = 2^4\pi\frac{3-5}{15} = -\frac{32\pi}{15}$$

$$\bar{z} = \frac{z-mom}{M} = -\frac{32\pi}{15}\frac{3}{8\pi} = -\frac{4}{5}$$

6. $\big(30$ points$\big)$ Do this problem, if you did the Interpretation of Div and Curl Project.

Find the divergence of the vector field $\vec{F} = (xz^2, yz^2, 0)$ at the point $(x,y,z) = (0,0,c)$.

a. by using the derivative defintion:

$$\vec{\nabla} \cdot \vec{F} = z^2 + z^2 = 2z^2 \qquad \vec{\nabla} \cdot \vec{F}\Big|_{(0,0,c)} = 2c^2$$

b. by using the integral definition:

HINTS: For a sphere of radius ρ centered at (a,b,c), if you use standard spherical coordinates, the normal vector is

$$\vec{N} = \left(\rho^2 \sin^2\varphi \cos\theta, \rho^2 \sin^2\varphi \sin\theta, \rho^2 \cos\varphi \sin\varphi\right)$$

The φ-integral can be done using the substitution $u = \cos\varphi$.

You can ignore terms in the integral proportional to ρ^n with $n > 3$ since they drop out of the limit.

$$\vec{\nabla} \cdot \vec{F}\Big|_{(0,0,c)} = \lim_{\rho \to 0} \frac{3}{4\pi\rho^3} \iint \vec{F} \cdot d\vec{S}$$

where the integral is over the sphere of radius ρ centered at $(0,0,c)$.

$$\vec{R}(\varphi,\theta) = (\rho \sin\varphi \cos\theta, \rho \sin\varphi \sin\theta, c + \rho\cos\varphi)$$

$$\vec{N} = \left(\rho^2 \sin^2\varphi \cos\theta, \rho^2 \sin^2\varphi \sin\theta, \rho^2 \cos\varphi \sin\varphi\right) \qquad \text{(Given)}$$

$$\vec{F} = (xz^2, yz^2, 0) = \left(\rho \sin\varphi \cos\theta (c + \rho\cos\varphi)^2, \rho \sin\varphi \sin\theta (c + \rho\cos\varphi)^2, 0\right)$$

$$\vec{F} \cdot \vec{N} = \rho^2 \sin^2\varphi \cos\theta \rho \sin\varphi \cos\theta (c + \rho\cos\varphi)^2 + \rho^2 \sin^2\varphi \sin\theta \rho \sin\varphi \sin\theta (c + \rho\cos\varphi)^2$$
$$= \rho^3 \sin^3\varphi (c + \rho\cos\varphi)^2$$

$$\iint \vec{F} \cdot d\vec{S} = \int_0^\pi \int_0^{2\pi} \vec{F} \cdot \vec{N}\, d\theta\, d\varphi = \int_0^\pi \int_0^{2\pi} \rho^3 \sin^3\varphi (c + \rho\cos\varphi)^2\, d\theta\, d\varphi = 2\pi\rho^3 \int_0^\pi \sin^3\varphi (c + \rho\cos\varphi)^2\, d\varphi$$

Drop terms of order greater than ρ^3 :

$$\iint \vec{F} \cdot d\vec{S} \approx 2\pi\rho^3 \int_0^\pi c^2 \sin^3\varphi\, d\varphi = 2\pi\rho^3 c^2 \int_0^\pi (1 - \cos^2\varphi) \sin\varphi\, d\varphi \qquad u = \cos\varphi \qquad du = -\sin\varphi\, d\varphi$$

$$\iint \vec{F} \cdot d\vec{S} \approx -2\pi\rho^3 c^2 \int_1^{-1} (1 - u^2)\, du = -2\pi\rho^3 c^2 \left[u - \frac{u^3}{3}\right]_1^{-1} = -2\pi\rho^3 c^2 \left[-\frac{2}{3}\right] + 2\pi\rho^3 c^2 \left[\frac{2}{3}\right] = \frac{8\pi\rho^3 c^2}{3}$$

$$\vec{\nabla} \cdot \vec{F}\Big|_{(0,0,c)} = \lim_{\rho \to 0} \frac{3}{4\pi\rho^3} \left(\frac{8\pi\rho^3 c^2}{3}\right) = 2c^2$$

7. $\left(30 \text{ points}\right)$ Do this problem, if you did the Gauss' and Ampere's Laws Project.

Find the total charge in the cylinder $x^2 + y^2 \le a^2, \quad 0 \le z \le 1$ if the electric field is

$$\vec{E} = \frac{\hat{r}}{r} = \frac{\vec{r}}{r^2} = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}, 0\right)$$

where $\vec{r} = (x, y, 0)$ and $r = \sqrt{x^2+y^2}$.

a. using the derivative form of Gauss' Law.

$$\rho = \frac{1}{4\pi}\vec{\nabla}\cdot\vec{E} = \frac{1}{4\pi}\left[\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)\right]$$

$$= \frac{1}{4\pi}\left[\frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} + \frac{(x^2+y^2) - y(2y)}{(x^2+y^2)^2}\right]$$

$$= \frac{1}{4\pi}\left[\frac{2(x^2+y^2) - 2x^2 - 2y^2}{(x^2+y^2)^2}\right] = 0$$

$$Q = \iiint_C \rho\, dV = 0$$

b. using the integral form of Gauss' Law.

$$4\pi Q = \iint_{\partial C} \vec{E}\cdot d\vec{S} = \iint_{top} \vec{E}\cdot d\vec{S} + \iint_{bottom} \vec{E}\cdot d\vec{S} + \iint_{sides} \vec{E}\cdot d\vec{S}$$

On the ends of the cylinder, the normal is $\vec{N} = \pm\hat{k}$ while $\vec{E}$ is horizontal. So $\vec{E}\cdot\vec{N} = 0$ and

$$\iint_{top} \vec{E}\cdot d\vec{S} = \iint_{bottom} \vec{E}\cdot d\vec{S} = 0$$

The sides are parametrized by $R(\theta, z) = (a\cos\theta, a\sin\theta, z)$.

$$\vec{R}_\theta = \left(-a\sin\theta, a\cos\theta, \ 0\right)$$
$$\vec{R}_z = (\ 0, \quad 0, \quad 1)$$
$$\vec{N} = \vec{R}_\theta \times \vec{R}_z = (a\cos\theta, a\sin\theta, 0)$$

$$\vec{E} = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}, 0\right) = \left(\frac{a\cos\theta}{a^2}, \frac{a\sin\theta}{a^2}, 0\right) = \left(\frac{\cos\theta}{a}, \frac{\sin\theta}{a}, 0\right)$$

$$\vec{E}\cdot\vec{N} = \cos^2\theta + \sin^2\theta = 1$$

$$4\pi Q = \iint_{sides} \vec{E}\cdot d\vec{S} = \int\int \vec{E}\cdot\vec{N}\, d\theta\, dz = \int_0^1\int_0^{2\pi} 1\, d\theta\, dz = 2\pi \qquad Q = \frac{1}{2}$$

c. What do these results tell you about the location of the electric charge? Why?

Part (a) says $\rho = 0$. So there is no charge wherever $\vec{E}$ and $\vec{\nabla}\cdot\vec{E}$ are defined which is everywhere but $r = 0$ which is the z-axis. However, part (b) says $Q = \frac{1}{2}$. So there must be charge along the z-axis.