Name________________________ ID____________

1	/15	4	/15
2	/15	5	/20
3	/15	6	/20

MATH 311 Exam 1 Fall 2002
Section 200 Solutions P. Yasskin

1. (15 points) Compute

$$\det\begin{pmatrix} 2 & 5 & 4 & -1 & 4 \\ 0 & 1 & -2 & 1 & 2 \\ 1 & 3 & 0 & -2 & 1 \\ 2 & 6 & 3 & -4 & -1 \\ -2 & -6 & -3 & 0 & 1 \end{pmatrix}$$

$$\begin{vmatrix} 2 & 5 & 4 & -1 & 4 \\ 0 & 1 & -2 & 1 & 2 \\ 1 & 3 & 0 & -2 & 1 \\ 2 & 6 & 3 & -4 & -1 \\ -2 & -6 & -3 & 0 & 1 \end{vmatrix}\begin{matrix} R_3 \\ \\ R_1 \\ \\ R_5+R_4 \end{matrix} = -\begin{vmatrix} 1 & 3 & 0 & -2 & 1 \\ 0 & 1 & -2 & 1 & 2 \\ 2 & 5 & 4 & -1 & 4 \\ 2 & 6 & 3 & -4 & -1 \\ 0 & 0 & 0 & -4 & 0 \end{vmatrix}\begin{matrix} \\ \\ R_3-2R_1 \\ R_4-2R_1 \\ \\ \end{matrix} = -\begin{vmatrix} 1 & 3 & 0 & -2 & 1 \\ 0 & 1 & -2 & 1 & 2 \\ 0 & -1 & 4 & 3 & 2 \\ 0 & 0 & 3 & 0 & -3 \\ 0 & 0 & 0 & -4 & 0 \end{vmatrix}\begin{matrix} \\ \\ R_3+R_2 \\ R_4/3 \\ \\ \end{matrix}$$

$$= -3\begin{vmatrix} 1 & 3 & 0 & -2 & 1 \\ 0 & 1 & -2 & 1 & 2 \\ 0 & 0 & 2 & 4 & 4 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & -4 & 0 \end{vmatrix}\begin{matrix} \\ \\ R_3/2 \\ \\ \\ \end{matrix} = -6\begin{vmatrix} 1 & 3 & 0 & -2 & 1 \\ 0 & 1 & -2 & 1 & 2 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & -4 & 0 \end{vmatrix}\begin{matrix} \\ \\ \\ R_4-R_3 \\ \\ \end{matrix} = -6\begin{vmatrix} 1 & 3 & 0 & -2 & 1 \\ 0 & 1 & -2 & 1 & 2 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & -4 & 0 \end{vmatrix}\begin{matrix} \\ \\ \\ R_4-R_3 \\ \\ \end{matrix}$$

$$= -6\begin{vmatrix} 1 & 3 & 0 & -2 & 1 \\ 0 & 1 & -2 & 1 & 2 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & -2 & -3 \\ 0 & 0 & 0 & -4 & 0 \end{vmatrix}\begin{matrix} \\ \\ \\ \\ R_5-2R_4 \end{matrix} = -6\begin{vmatrix} 1 & 3 & 0 & -2 & 1 \\ 0 & 1 & -2 & 1 & 2 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & -2 & -3 \\ 0 & 0 & 0 & 0 & 6 \end{vmatrix} = -6(-2)6 = 72$$

2. (15 points) Use row operations on the augmented matrix to solve the system of equations

$$2u + 4v - 2w - 2x + 2y - 4z = 4$$
$$u + 2v - 2w - 3x + y - 3z = -3$$
$$3u + 6v - 2w - x + 3y - 5z = b$$

$$\left(\begin{array}{cccccc|c} 2 & 4 & -2 & -2 & 2 & -4 & 4 \\ 1 & 2 & -2 & -3 & 1 & -3 & -3 \\ 3 & 6 & -2 & -1 & 3 & -5 & b \end{array}\right) \begin{array}{l} R_2 \\ R_1 \\ \end{array} \Rightarrow \left(\begin{array}{cccccc|c} 1 & 2 & -2 & -3 & 1 & -3 & -3 \\ 2 & 4 & -2 & -2 & 2 & -4 & 4 \\ 3 & 6 & -2 & -1 & 3 & -5 & b \end{array}\right) \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

$$\Rightarrow \left(\begin{array}{cccccc|c} 1 & 2 & -2 & -3 & 1 & -3 & -3 \\ 0 & 0 & 2 & 4 & 0 & 2 & 10 \\ 0 & 0 & 4 & 8 & 0 & 4 & b+9 \end{array}\right) \begin{array}{l} R_1 + R_2 \\ R_2/2 \\ R_3 - 2R_2 \end{array} \Rightarrow \left(\begin{array}{cccccc|c} 1 & 2 & 0 & 1 & 1 & -1 & 7 \\ 0 & 0 & 1 & 2 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & b-11 \end{array}\right)$$

a. For what value(s) of b do there exist solutions.

$b = 11$

b. For those value(s) of b write the set of all solutions in parametric form.

u and w are leading variables. v, x, y and z are free variables.

$u = 7 - 2p - r - s + t$ $\qquad x = r$

$v = p$ $\qquad y = s$

$w = 5 - 2r - t$ $\qquad z = t$

c. Interpret the solution set as a k-plane in $\mathbb{R}^n$ for some k and n.

4-plane in $\mathbb{R}^6$

3. (15 points) Find the equation of the plane tangent to the graph of the function $f(x,y) = 3x\sin y - 2y\cos x$ at the point $(x,y) = \left(0, \frac{\pi}{2}\right)$.

$f(x,y) = 3x\sin y - 2y\cos x \qquad f\left(0, \frac{\pi}{2}\right) = 0 - 2\frac{\pi}{2}\cos 0 = -\pi$

$f_x(x,y) = 3\sin y + 2y\sin x \qquad f_x\left(0, \frac{\pi}{2}\right) = 3\sin\frac{\pi}{2} + 0 = 3$

$f_y(x,y) = 3x\cos y - 2\cos x \qquad f_y\left(0, \frac{\pi}{2}\right) = 0 - 2\cos 0 = -2$

$$z = f\left(0, \frac{\pi}{2}\right) + f_x\left(0, \frac{\pi}{2}\right)(x - 0) + f_y\left(0, \frac{\pi}{2}\right)\left(y - \frac{\pi}{2}\right)$$

$z = -\pi + 3(x) - 2\left(y - \frac{\pi}{2}\right)$ or $z = 3x - 2y$

4. (15 points) Find the equation of the line perpendicular to the surface $F(x,y,z) = x^2y + y^3z + z^4x = 29$ at the point $P = (x,y,z) = (3,2,1)$.

$$\vec{\nabla}F = (2xy + z^4, x^2 + 3y^2z, y^3 + 4z^3x)$$

$$\vec{v} = \vec{\nabla}F\Big|_{(3,2,1)} = (2 \cdot 3 \cdot 2 + 1^4, 3^2 + 3 \cdot 2^2 \cdot 1, 2^3 + 4 \cdot 1^3 \cdot 3) = (13,21,20)$$

$$X = P + t\vec{v} \qquad (x,y,z) = (3,2,1) + t(13,21,20) \qquad \begin{aligned} x &= 3 + 13t \\ y &= 2 + 21t \\ z &= 1 + 20t \end{aligned}$$

5. (20 points) Consider the matrix $A = \begin{pmatrix} -2 & -5 & 1 \\ 1 & 3 & -1 \\ 3 & 7 & -1 \end{pmatrix}$.

a. Find A^{-1} or show it does not exist.

$$\left(\begin{array}{ccc|ccc} -2 & -5 & 1 & 1 & 0 & 0 \\ 1 & 3 & -1 & 0 & 1 & 0 \\ 3 & 7 & -1 & 0 & 0 & 1 \end{array}\right) \begin{matrix} R_2 \\ R_1 \\ \end{matrix} \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 3 & -1 & 0 & 1 & 0 \\ -2 & -5 & 1 & 1 & 0 & 0 \\ 3 & 7 & -1 & 0 & 0 & 1 \end{array}\right) \begin{matrix} \\ R_2 + 2R_1 \\ R_3 - 3R_1 \end{matrix}$$

$$\Rightarrow \left(\begin{array}{ccc|ccc} 1 & 3 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 2 & 0 \\ 0 & -2 & 2 & 0 & -3 & 1 \end{array}\right) \begin{matrix} R_1 - 3R_2 \\ \\ R_3 + 2R_2 \end{matrix} \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & -3 & -5 & 0 \\ 0 & 1 & -1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 2 & 1 & 1 \end{array}\right)$$

A^{-1} does not exist because there is a row of zeros.

b. Consider the equations $AX = \mathbf{0}$, i.e.

$$\begin{aligned} -2x - 5y + z &= 0 \\ x + 3y - z &= 0 \\ 3x + 7y - z &= 0 \end{aligned}$$

How many solutions are there?
Circle one: (Do not solve the equations.)

No Solutions   Unique Solution   **[∞-Many Solutions]**

Explain why:

The equations are homogeneous. So there is at least one solution: $(0,0,0)$.
A is not invertible. So there is not a unique solution.

6. (20 points) Consider the matrix $A = \begin{pmatrix} -2 & -5 & 1 \\ 1 & 3 & -1 \\ 3 & 7 & 0 \end{pmatrix}$.

a. Find A^{-1} or show it does not exist.

$$\left(\begin{array}{ccc|ccc} -2 & -5 & 1 & 1 & 0 & 0 \\ 1 & 3 & -1 & 0 & 1 & 0 \\ 3 & 7 & 0 & 0 & 0 & 1 \end{array}\right) \begin{matrix} R_2 \\ R_1 \\ \end{matrix} \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 3 & -1 & 0 & 1 & 0 \\ -2 & -5 & 1 & 1 & 0 & 0 \\ 3 & 7 & 0 & 0 & 0 & 1 \end{array}\right) \begin{matrix} \\ R_2 + 2R_1 \\ R_3 - 3R_1 \end{matrix}$$

$$\Rightarrow \left(\begin{array}{ccc|ccc} 1 & 3 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 2 & 0 \\ 0 & -2 & 3 & 0 & -3 & 1 \end{array}\right) \begin{matrix} R_1 - 3R_2 \\ \\ R_3 + 2R_2 \end{matrix} \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & -3 & -5 & 0 \\ 0 & 1 & -1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 & 1 & 1 \end{array}\right) \begin{matrix} R_1 - 2R_3 \\ R_2 + R_3 \\ \end{matrix}$$

$$\Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -7 & -7 & -2 \\ 0 & 1 & 0 & 3 & 3 & 1 \\ 0 & 0 & 1 & 2 & 1 & 1 \end{array}\right) \quad \text{So} \quad A^{-1} = \begin{pmatrix} -7 & -7 & -2 \\ 3 & 3 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$

Check: $$\begin{pmatrix} -2 & -5 & 1 \\ 1 & 3 & -1 \\ 3 & 7 & 0 \end{pmatrix}\begin{pmatrix} -7 & -7 & -2 \\ 3 & 3 & 1 \\ 2 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

b. Consider the equations $AX = \hat{\jmath}$, i.e.

$$\begin{aligned} -2x - 5y + z &= 0 \\ x + 3y - z &= 1 \\ 3x + 7y \qquad &= 0 \end{aligned}$$

How many solutions are there?
Circle one:

No Solutions | **[Unique Solution]** | ∞-Many Solutions

Find all solutions if there are any.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = X = A^{-1}\hat{\jmath} = \begin{pmatrix} -7 & -7 & -2 \\ 3 & 3 & 1 \\ 2 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -7 \\ 3 \\ 1 \end{pmatrix}$$