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MATH 311 Exam 2 Fall 2002

Section 200 P. Yasskin

1	/10	3	/30
2	/20	4	/45

1. $(10$ points$)$ A linear map $f :$ **R**$^p \rightarrow$ **R**^{q} has matrix $A = \begin{pmatrix} 3 & 0 \\ 2 & -1 \\ 0 & 3 \end{pmatrix}$ and

a linear map $g :$ **R**$^q \rightarrow$ **R**^{p} has matrix $B = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 2 \end{pmatrix}$.

a. (4) What are p and q ?

b. (2) In the composition $g \circ f :$ **R**$^n \rightarrow$ **R**^{n}, what is n ?

c. (4) What is the matrix of $g \circ f$?

2. (20 points) Consider the vector space $V = Span\{p_1, \ p_2, \ p_3, \ p_4\}$ where

$$p_1 = 1 + 2x - x^3, \quad p_2 = 2 + 4x + x^4, \quad p_3 = 3 + 6x - x^3 + x^4, \quad p_4 = 2x^3 + x^4$$

Pare $\{p_1, \ p_2, \ p_3, \ p_4\}$ down to a basis for V. (Don't bother proving the final set is a basis.) What is $\dim V$?

3. (30 points) Consider the curvilinear coordinate system $(x,y) = \vec{R}(u,v) = \left(uv, \frac{u}{v}\right)$, i.e.

$$x = uv \qquad y = \frac{u}{v}$$

a. (5) Describe the u-coordinate curve for which $v = 2$.
(Give an xy-equation and describe the shape in words.)

b. (6) Find $\vec{e}_u$, the vector tangent to the u-curve at the point $(u,v) = (1,2)$.

c. (5) Describe the v-coordinate curve for which $u = 1$.
(Give an xy-equation and describe the shape in words.)

d. (6) Find $\vec{e}_v$, the vector tangent to the v-curve at the point $(u,v) = (1,2)$.

e. (8) Let P be the pressure in a gas.
Let $\vec{\nabla}P = \left(\frac{\partial P}{\partial x}, \frac{\partial P}{\partial y}\right)$ be its gradient in rectangular coordinates and
let $\vec{\nabla}\left(P \circ \vec{R}\right) = \left(\frac{\partial\left(P \circ \vec{R}\right)}{\partial u}, \frac{\partial\left(P \circ \vec{R}\right)}{\partial v}\right)$ be its gradient in the u,v-curvilinear coordinates.
If $\vec{\nabla}P\Big|_{(x,y)=(2,1/2)} = (16,20)$, find $\vec{\nabla}\left(P \circ \vec{R}\right)\Big|_{(u,v)=(1,2)}$. HINT: Use the chain rule.

4. (40 points + 5 Extra Credit) Consider the vector spaces $V = Span\{\sinh x, \cosh x\}$ and $M(2,2) = \{2 \times 2 \text{ matrices}\}$. Consider two bases on V:

$$\{h_1 = \sinh x, \quad h_2 = \cosh x\} \qquad \text{and} \qquad \{e_1 = e^x, \quad e_2 = e^{-x}\}$$

Consider two bases on $M(2,2)$:

$$\left\{ m_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad m_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad m_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad m_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

and

$$\left\{ n_1 = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \quad n_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad n_3 = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}, \quad n_4 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\}$$

Consider the linear map $L : V \rightarrow M(2,2)$ given by

$$L(f) = \begin{pmatrix} f(0) & f'(0) \\ f(\ln 2) & f'(\ln 2) \end{pmatrix}$$

Note: $e^{\ln 2} = 2, \qquad e^{-\ln 2} = \frac{1}{2}, \qquad \sinh(\ln 2) = \frac{3}{4}, \qquad \cosh(\ln 2) = \frac{5}{4}$

a. (2) Identify the domain of L and its dimension.

b. (2) Identify the codomain of L and its dimension.

c. (4) Is the function L one-to-one? Why? HINT: Let $f = ae^x + be^{-x}$ and $g = ce^x + de^{-x}$.

Recall: $L(f) = \begin{pmatrix} f(0) & f'(0) \\ f(\ln 2) & f'(\ln 2) \end{pmatrix}$

d. $(2+5$ E.C. $)$ Find the Image of L. Then express it as the Span of some matrices (with constant entries). What is its dimension?

e. (4) Is the function L onto? Why?

Recall: $L(f) = \begin{pmatrix} f(0) & f'(0) \\ f(\ln 2) & f'(\ln 2) \end{pmatrix}$

f. (4) Find the matrix of L from the h basis to the m basis. (Call it $\underset{m\leftarrow h}{A}$.)

g. (4) Find the matrix of L from the e basis to the n basis. (Call it $\underset{n\leftarrow e}{B}$.)
Use the definitions of L and B, not the change of basis matrices.

h. (6) Find the change of basis matrices between the e and h bases. (Call them $\underset{h\leftarrow e}{C}$ and $\underset{e\leftarrow h}{C}$.)
Be sure to identify which is which!

Recall: $L(f) = \begin{pmatrix} f(0) & f'(0) \\ f(\ln 2) & f'(\ln 2) \end{pmatrix}$

i. (0) The change of basis matrices between the m and n bases are:

$$\underset{m\leftarrow n}{C} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \quad \text{and} \quad \underset{n\leftarrow m}{C} = \underset{m\leftarrow n}{C}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

These are given. Do not compute them!

j. (4) Recompute $\underset{n\leftarrow e}{B}$, the matrix of L from the e basis to the n basis by using the change of basis matrices.

Recall: $L(f) = \begin{pmatrix} f(0) & f'(0) \\ f(\ln 2) & f'(\ln 2) \end{pmatrix}$

k. (2) For the function $f = 6e^x + 4e^{-x}$, compute $L(f)$ from the definition of L.

l. (3) For the function $f = 6e^x + 4e^{-x}$, compute $(f)_e$ and $(f)_h$ which are the components of f relative to the e and h bases, respectively. Check $(f)_h$ by hooking the components onto the basis.

m. (3) For the function $f = 6e^x + 4e^{-x}$, compute $[L(f)]_n$ and check by hooking the components onto the basis.