Name________________________ ID____________

1	/10	3	/30
2	/20	4	/45

MATH 311 Exam 2 Fall 2002
Section 200 Solutions P. Yasskin

1. $\left(10 \text{ points}\right)$ A linear map $f : \mathbf{R}^p \to \mathbf{R}^q$ has matrix $A = \begin{pmatrix} 3 & 0 \\ 2 & -1 \\ 0 & 3 \end{pmatrix}$ and

a linear map $g : \mathbf{R}^q \to \mathbf{R}^p$ has matrix $B = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 2 \end{pmatrix}$.

a. (4) What are p and q ?

A is 3×2. So $q = 3$ and $p = 2$.

b. (2) In the composition $g \circ f : \mathbf{R}^n \to \mathbf{R}^n$, what is n ?

In $g \circ f$, the map f acts first. So $n = p = 2$.

c. (4) What is the matrix of $g \circ f$?

$\vec{y} = (g \circ f)(\vec{x}) = g(f(\vec{x}))$ means $\vec{y} = g(\vec{z}) = B\vec{z}$ where $\vec{z} = f(\vec{x}) = A\vec{x}$.

So $\vec{y} = B\vec{z} = BA\vec{x}$. Thus the matrix of $g \circ f$ is $BA = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 2 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 2 & -1 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 6 & 3 \\ 6 & 3 \end{pmatrix}$

2. $\left(20 \text{ points}\right)$ Consider the vector space $V = Span\{p_1, \ p_2, \ p_3, \ p_4\}$ where

$$p_1 = 1 + 2x - x^3, \quad p_2 = 2 + 4x + x^4, \quad p_3 = 3 + 6x - x^3 + x^4, \quad p_4 = 2x^3 + x^4$$

Pare $\{p_1, \ p_2, \ p_3, \ p_4\}$ down to a basis for V. (Don't bother proving the final set is a basis.) What is $\dim V$?

Are $p_1, \ p_2, \ p_3, \ p_4$ linearly independent? Assume $ap_1 + bp_2 + cp_3 + dp_4 = 0$.
$a(1 + 2x - x^3) + b(2 + 4x + x^4) + c(3 + 6x - x^3 + x^4) + d(2x^3 + x^4) = 0$

$$\begin{aligned} a + 2b + 3c &= 0 \\ 2a + 4b + 6c &= 0 \\ -a - c + 2d &= 0 \\ b + c + d &= 0 \end{aligned} \quad \Rightarrow \quad \left(\begin{array}{cccc|c} 1 & 2 & 3 & 0 & 0 \\ 2 & 4 & 6 & 0 & 0 \\ -1 & 0 & -1 & 2 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{array}\right) \begin{matrix} \\ R_2 - 2R_1 \\ R_3 + R_1 \\ \\ \end{matrix} \quad \Rightarrow$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 2 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{array}\right) \begin{matrix} R_1 - 2R_4 \\ \\ R_3 - 2R_4 \\ \\ \end{matrix} \quad \Rightarrow \quad \left(\begin{array}{cccc|c} 1 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{array}\right) \quad \Rightarrow \quad \begin{aligned} a &= -s + 2t \\ b &= -s - t \\ c &= s \\ d &= t \end{aligned}$$

c and d are free variables. So p_3 and p_4 can be solved for, leaving p_1 and p_2 as the basis.
$\dim V = 2$.

3. $(30$ points$)$ Consider the curvilinear coordinate system $(x,y) = \vec{R}(u,v) = \left(uv, \frac{u}{v}\right)$, i.e.

$$x = uv \qquad y = \frac{u}{v}$$

a. (5) Describe the u-coordinate curve for which $v = 2$.
(Give an xy-equation and describe the shape in words.)

If $v = 2$, then $x = 2u$, $y = \frac{u}{2}$. So $u = \frac{x}{2}$ and $y = \frac{x}{4}$.
This is the line thru the origin with slope $\frac{1}{4}$.

b. (6) Find $\vec{e}_u$, the vector tangent to the u-curve at the point $(u,v) = (1,2)$.

$$\vec{e}_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}\right) = \left(v, \frac{1}{v}\right) \qquad \vec{e}_u\big|_{(1,2)} = \left(2, \frac{1}{2}\right)$$

c. (5) Describe the v-coordinate curve for which $u = 1$.
(Give an xy-equation and describe the shape in words.)

If $u = 1$, then $x = v$, $y = \frac{1}{v}$. So $y = \frac{1}{x}$.
This is a hyperbola in the first and third quadrants.

d. (6) Find $\vec{e}_v$, the vector tangent to the v-curve at the point $(u,v) = (1,2)$.

$$\vec{e}_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}\right) = \left(u, \frac{-u}{v^2}\right) \qquad \vec{e}_v\big|_{(1,2)} = \left(1, \frac{-1}{4}\right)$$

e. (8) Let P be the pressure in a gas.
Let $\vec{\nabla}P = \left(\frac{\partial P}{\partial x}, \frac{\partial P}{\partial y}\right)$ be its gradient in rectangular coordinates and
let $\vec{\nabla}\left(P \circ \vec{R}\right) = \left(\frac{\partial\left(P \circ \vec{R}\right)}{\partial u}, \frac{\partial\left(P \circ \vec{R}\right)}{\partial v}\right)$ be its gradient in the u,v-curvilinear coordinates.
If $\vec{\nabla}P\Big|_{(x,y)=(2,1/2)} = (16,20)$, find $\vec{\nabla}\left(P \circ \vec{R}\right)\Big|_{(u,v)=(1,2)}$. HINT: Use the chain rule.

By the chain rule: $\vec{\nabla}\left(P \circ \vec{R}\right) = \vec{\nabla}P\, J\vec{R} = \left(\frac{\partial P}{\partial x}, \frac{\partial P}{\partial y}\right)\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$

Note: The columns of the Jacobian are the vectors $\vec{e}_u$ and $\vec{e}_v$.
Further, when $(u,v) = (1,2)$, we have $(x,y) = \left(uv, \frac{u}{v}\right) = \left(2, \frac{1}{2}\right)$. So

$$\vec{\nabla}\left(P \circ \vec{R}\right)\Big|_{(u,v)=(1,2)} = \vec{\nabla}P\Big|_{(x,y)=(2,1/2)} J\vec{R}\Big|_{(u,v)=(1,2)} = (16,20)\begin{pmatrix} 2 & 1 \\ \frac{1}{2} & \frac{-1}{4} \end{pmatrix} = (42,11)$$

4. $\left(40 \text{ points} + 5 \text{ Extra Credit}\right)$ Consider the vector spaces $V = Span\{\sinh x, \cosh x\}$ and $M(2,2) = \left\{2 \times 2 \text{ matrices}\right\}$. Consider two bases on V:

$$\{h_1 = \sinh x, \quad h_2 = \cosh x\} \qquad \text{and} \qquad \{e_1 = e^x, \quad e_2 = e^{-x}\}$$

Consider two bases on $M(2,2)$:

$$\left\{m_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad m_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad m_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad m_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$$

and

$$\left\{n_1 = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \quad n_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad n_3 = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}, \quad n_4 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\right\}$$

Consider the linear map $L : V \to M(2,2)$ given by

$$L(f) = \begin{pmatrix} f(0) & f'(0) \\ f(\ln 2) & f'(\ln 2) \end{pmatrix}$$

Note: $e^{\ln 2} = 2, \qquad e^{-\ln 2} = \frac{1}{2}, \qquad \sinh(\ln 2) = \frac{3}{4}, \qquad \cosh(\ln 2) = \frac{5}{4}$

a. (2) Identify the domain of L and its dimension.

$Dom(L) = V \qquad \dim Dom(L) = 2$

b. (2) Identify the codomain of L and its dimension.

$Codom(L) = M(2,2) \qquad \dim Codom(L) = 4$

c. (4) Is the function L one-to-one? Why? HINT: Let $f = ae^x + be^{-x}$ and $g = ce^x + de^{-x}$.

$$L(f) = \begin{pmatrix} a+b & a-b \\ 2a + \frac{b}{2} & 2a - \frac{b}{2} \end{pmatrix} \qquad L(g) = \begin{pmatrix} c+d & c-d \\ 2c + \frac{d}{2} & 2c - \frac{d}{2} \end{pmatrix}$$

$$L(f) = L(g) \quad \Rightarrow \quad \begin{matrix} a+b & c+d \\ a-b & c-d \\ 2a + \frac{b}{2} & 2c + \frac{d}{2} \\ 2a - \frac{b}{2} & 2c - \frac{d}{2} \end{matrix} \quad \Rightarrow \quad \begin{matrix} a = c \\ b = d \end{matrix} \quad \Rightarrow \quad f = g$$

So L is one-to-one.

d. $\left(2 + 5 \text{ E.C.}\right)$ Find the Image of L. Then express it as the Span of some matrices (with constant entries). What is its dimension?

$$\mathrm{Im}(L) = \{L(f)\} = \left\{\begin{pmatrix} a+b & a-b \\ 2a + \frac{b}{2} & 2a - \frac{b}{2} \end{pmatrix}\right\}$$

$$= \left\{a\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} + b\begin{pmatrix} 1 & -1 \\ \frac{1}{2} & \frac{-1}{2} \end{pmatrix}\right\} = Span\left\{\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ \frac{1}{2} & \frac{-1}{2} \end{pmatrix}\right\}$$

$\dim \mathrm{Im}(L) = 2$

e. (4) Is the function L onto? Why?

Easy Way: L is not onto because $\dim Codom(L) = 4$ but $\dim \operatorname{Im}(L) = 2$

Hard Way: Given $M = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in M(2,2)$, is there an $f = ae^x + be^{-x} \in V$ such that

$L(f) = M$? Given p,q,r,s, solve $\begin{pmatrix} a+b & a-b \\ 2a+\frac{b}{2} & 2a-\frac{b}{2} \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ for a,b.

$$\left(\begin{array}{cc|c} 1 & 1 & p \\ 1 & -1 & q \\ 2 & \frac{1}{2} & r \\ 2 & \frac{-1}{2} & s \end{array}\right) \Rightarrow \quad \cdots \quad \Rightarrow \left(\begin{array}{cc|c} 1 & 0 & \frac{p+q}{2} \\ 0 & 1 & \frac{p-q}{2} \\ 0 & 0 & r-\frac{3}{4}q-\frac{5}{4}p \\ 0 & 0 & s-\frac{5}{4}q-\frac{3}{4}p \end{array}\right)$$

No solution for general p,q,r,s. L is not onto.

f. (4) Find the matrix of L from the h basis to the m basis. (Call it $\underset{m\leftarrow h}{A}$.)

$$L(h_1) = L(\sinh x) = \begin{pmatrix} 0 & 1 \\ \frac{3}{4} & \frac{5}{4} \end{pmatrix} = m_2 + \frac{3}{4}m_3 + \frac{5}{4}m_4$$

$$L(h_2) = L(\cosh x) = \begin{pmatrix} 1 & 0 \\ \frac{5}{4} & \frac{3}{4} \end{pmatrix} = m_1 + \frac{5}{4}m_3 + \frac{3}{4}m_4$$

$$\Rightarrow \quad \underset{m\leftarrow h}{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ \frac{3}{4} & \frac{5}{4} \\ \frac{5}{4} & \frac{3}{4} \end{pmatrix}$$

g. (4) Find the matrix of L from the e basis to the n basis. (Call it $\underset{n\leftarrow e}{B}$.)

Use the definitions of L and B, not the change of basis matrices.

$$L(e_1) = L(e^x) = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} = n_2 + 2n_4$$

$$L(e_2) = L(e^{-x}) = \begin{pmatrix} 1 & -1 \\ \frac{1}{2} & \frac{-1}{2} \end{pmatrix} = n_1 + \frac{1}{2}n_3$$

$$\Rightarrow \quad \underset{n\leftarrow e}{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & \frac{1}{2} \\ 2 & 0 \end{pmatrix}$$

h. (6) Find the change of basis matrices between the e and h bases. (Call them $\underset{h\leftarrow e}{C}$ and $\underset{e\leftarrow h}{C}$.)

Be sure to identify which is which!

$$h_1 = \sinh x = \frac{e^x - e^{-x}}{2} = \frac{1}{2}e_1 - \frac{1}{2}e_2$$

$$h_2 = \cosh x = \frac{e^x + e^{-x}}{2} = \frac{1}{2}e_1 + \frac{1}{2}e_2$$

$$\Rightarrow \quad \underset{e\leftarrow h}{C} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$\Rightarrow \quad \underset{h\leftarrow e}{C} = \underset{e\leftarrow h}{C}^{-1} = \frac{1}{\frac{1}{4}+\frac{1}{4}} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

i. (0) The change of basis matrices between the m and n bases are:

$$\underset{m\leftarrow n}{C} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \quad \text{and} \quad \underset{n\leftarrow m}{C} = \underset{m\leftarrow n}{C}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

These are given. Do not compute them!

j. (4) Recompute $\underset{n\leftarrow e}{B}$, the matrix of L from the e basis to the n basis by using the change of basis matrices.

$$\underset{n\leftarrow e}{B} = \underset{n\leftarrow m}{C}\ \underset{m\leftarrow h}{A}\ \underset{h\leftarrow e}{C} = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \\ \frac{3}{4} & \frac{5}{4} \\ \frac{5}{4} & \frac{3}{4} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \cdots = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & \frac{1}{2} \\ 2 & 0 \end{pmatrix}$$

k. (2) For the function $f = 6e^{x} + 4e^{-x}$, compute $L(f)$ from the definition of L.

$$L(f) = \begin{pmatrix} f(0) & f'(0) \\ f(\ln 2) & f'(\ln 2) \end{pmatrix} = \begin{pmatrix} 10 & 2 \\ 14 & 10 \end{pmatrix}$$

l. (3) For the function $f = 6e^{x} + 4e^{-x}$, compute $(f)_e$ and $(f)_h$ which are the components of f relative to the e and h bases, respectively. Check $(f)_h$ by hooking the components onto the basis.

$$(f)_e = \begin{pmatrix} 6 \\ 4 \end{pmatrix} \qquad (f)_h = \underset{h\leftarrow e}{C}\,(f)_e = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 6 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 10 \end{pmatrix}$$

$$f = 2h_1 + 10h_2 = 2\sinh x + 10\cosh x = 2\left(\frac{e^{x} - e^{-x}}{2}\right) + 10\left(\frac{e^{x} + e^{-x}}{2}\right) = 6e^{x} + 4e^{-x}$$

m. (3) For the function $f = 6e^{x} + 4e^{-x}$, compute $[L(f)]_n$ and check by hooking the components onto the basis.

$$[L(f)]_n = \underset{n\leftarrow e}{B}\,(f)_e = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & \frac{1}{2} \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 6 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 2 \\ 12 \end{pmatrix}$$

$$L(f) = 4n_1 + 6n_2 + 2n_3 + 12n_4 = 4\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} + 6\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} + 12\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$$