Name________________________ ID___________

1	/10	4	/15
2	/10	5	/25
3	/20	6	/20

MATH 311 Exam 1 Spring 2003
Section 200 Solutions P. Yasskin

If you use row or column operations, be sure to give your reasons.

1. (10 points) Find a parametric equation for the line tangent to the curve $\vec{r}(t) = (t, t^2, t^3)$ at the point where $t = 2$.

$P = \vec{r}(2) = (2,4,8) \qquad \vec{v}(t) = (1, 2t, 3t^2) \qquad \vec{v}(2) = (1,4,12)$

$X = P + t\vec{v} = (2,4,8) + t(1,4,12) \qquad (x,y,z) = (2+t, 4+4t, 8+12t)$

OR

$X = P + (t-2)\vec{v} = (2,4,8) + (t-2)(1,4,12) \qquad (x,y,z) = (t, -4+4t, -16+12t)$

2. (10 points) Find the non-parametric equation for the plane tangent to the surface $x^3y^2 + xz^3 = 31$ at the point $(x,y,z) = (1,2,3)$.

$f = x^3y^2 + xz^3 \qquad \vec{\nabla} f = \left(3x^2y^2 + z^3,\ 2x^3y,\ 3xz^2\right)$

$\vec{N} = \vec{\nabla} f\Big|_{(1,2,3)} = \left(3 \cdot 2^2 + 3^3,\ 2 \cdot 2,\ 3 \cdot 3^2\right) = (39,4,27)$

$\vec{N} \cdot X = \vec{N} \cdot P \quad \Rightarrow \quad (39,4,27) \cdot (x,y,z) = (39,4,27) \cdot (1,2,3)$

$39x + 4y + 27z = 128$

3. (20 points) The following is a parametric surface in $\mathbb{R}^4$:

$$(w,x,y,z) = \vec{R}(u,v) = \left(u\sqrt{2}\cos v,\ u\sqrt{2}\sin v,\ v\cos u,\ v\sin u\right)$$

a. Find the two tangent vectors $\vec{e}_u$ and $\vec{e}_v$ at the point where $(u,v) = \left(\frac{\pi}{2}, \frac{\pi}{4}\right)$.

$$\vec{e}_u = \left(\sqrt{2}\cos v,\ \sqrt{2}\sin v,\ -v\sin u,\ v\cos u\right)$$

$$\vec{e}_v = \left(-u\sqrt{2}\sin v,\ u\sqrt{2}\cos v,\ \cos u,\ \sin u\right)$$

$$\vec{e}_u\left(\frac{\pi}{2}, \frac{\pi}{4}\right) = \left(\sqrt{2}\cos\frac{\pi}{4},\ \sqrt{2}\sin\frac{\pi}{4},\ -\frac{\pi}{4}\sin\frac{\pi}{2},\ \frac{\pi}{4}\cos\frac{\pi}{2}\right) = \left(1,\ 1,\ -\frac{\pi}{4},\ 0\right)$$

$$\vec{e}_v\left(\frac{\pi}{2}, \frac{\pi}{4}\right) = \left(-\frac{\pi}{2}\sqrt{2}\sin\frac{\pi}{4},\ \frac{\pi}{2}\sqrt{2}\cos\frac{\pi}{4},\ \cos\frac{\pi}{2},\ \sin\frac{\pi}{2}\right)\left(-\frac{\pi}{2},\ \frac{\pi}{2},\ 0,\ 1\right)$$

b. Find a parametric equation for the plane tangent to the surface $\vec{R}(u,v)$ at the point where $(u,v) = \left(\frac{\pi}{2}, \frac{\pi}{4}\right)$.

$$\vec{R}\left(\frac{\pi}{2}, \frac{\pi}{4}\right) = \left(\frac{\pi}{2}\sqrt{2}\cos\frac{\pi}{4},\ \frac{\pi}{2}\sqrt{2}\sin\frac{\pi}{4},\ \frac{\pi}{4}\cos\frac{\pi}{2},\ \frac{\pi}{4}\sin\frac{\pi}{2}\right) = \left(\frac{\pi}{2},\ \frac{\pi}{2},\ 0,\ \frac{\pi}{4}\right)$$

$$X = \vec{R}\left(\frac{\pi}{2}, \frac{\pi}{4}\right) + u\,\vec{e}_u\left(\frac{\pi}{2}, \frac{\pi}{4}\right) + v\,\vec{e}_v\left(\frac{\pi}{2}, \frac{\pi}{4}\right)$$

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{\pi}{2} \\ \frac{\pi}{2} \\ 0 \\ \frac{\pi}{4} \end{pmatrix} + u\begin{pmatrix} 1 \\ 1 \\ -\frac{\pi}{4} \\ 0 \end{pmatrix} + v\begin{pmatrix} -\frac{\pi}{2} \\ \frac{\pi}{2} \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{\pi}{2} + u - \frac{\pi}{2}v \\ \frac{\pi}{2} + u + \frac{\pi}{2}v \\ -\frac{\pi}{4}u \\ \frac{\pi}{4} + v \end{pmatrix}$$

OR

$$X = \vec{R}\left(\frac{\pi}{2}, \frac{\pi}{4}\right) + \left(u - \frac{\pi}{2}\right)\vec{e}_u\left(\frac{\pi}{2}, \frac{\pi}{4}\right) + \left(v - \frac{\pi}{4}\right)\vec{e}_v\left(\frac{\pi}{2}, \frac{\pi}{4}\right)$$

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{\pi}{2} \\ \frac{\pi}{2} \\ 0 \\ \frac{\pi}{4} \end{pmatrix} + \left(u - \frac{\pi}{2}\right)\begin{pmatrix} 1 \\ 1 \\ -\frac{\pi}{4} \\ 0 \end{pmatrix} + \left(v - \frac{\pi}{4}\right)\begin{pmatrix} -\frac{\pi}{2} \\ \frac{\pi}{2} \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} u - \frac{\pi}{2}v + \frac{\pi^2}{8} \\ u + \frac{\pi}{2}v - \frac{\pi^2}{8} \\ -\frac{\pi}{4}u + \frac{\pi^2}{8} \\ v \end{pmatrix}$$

4. (15 points) Let

$$M = \begin{pmatrix} 2 & 5 & 4 & -1 \\ 0 & 1 & -2 & 1 \\ 1 & 3 & 0 & -2 \\ 2 & 6 & 3 & x \end{pmatrix}$$

a. Compute $\det M$ (as a function of x).

$$\det M = \begin{vmatrix} 2 & 5 & 4 & -1 \\ 0 & 1 & -2 & 1 \\ 1 & 3 & 0 & -2 \\ 2 & 6 & 3 & x \end{vmatrix} \begin{matrix} R_3 \\ \\ R_1 \\ \\ \end{matrix} = -\begin{vmatrix} 1 & 3 & 0 & -2 \\ 0 & 1 & -2 & 1 \\ 2 & 5 & 4 & -1 \\ 2 & 6 & 3 & x \end{vmatrix} \begin{matrix} \\ \\ R_3 - 2R_1 \\ R_4 - 2R_1 \end{matrix} = -\begin{vmatrix} 1 & 3 & 0 & -2 \\ 0 & 1 & -2 & 1 \\ 0 & -1 & 4 & 3 \\ 0 & 0 & 3 & x+4 \end{vmatrix} \begin{matrix} \\ \\ R_3 + R_2 \\ \\ \end{matrix}$$

$$= -\begin{vmatrix} 1 & 3 & 0 & -2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 3 & x+4 \end{vmatrix} \begin{matrix} \\ \\ R_3/2 \\ \\ \end{matrix} = -2\begin{vmatrix} 1 & 3 & 0 & -2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & x+4 \end{vmatrix} \begin{matrix} \\ \\ \\ R_4 - 3R_3 \end{matrix} = -2\begin{vmatrix} 1 & 3 & 0 & -2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & x-2 \end{vmatrix}$$

$$= -2(x-2)$$

b. For what value(s) of x does M^{-1} exist? Why?

M^{-1} exist $\Leftrightarrow$ $\det M \neq 0$ $\Leftrightarrow$ $x \neq 2$

5. (25 points) Let $A = \begin{pmatrix} 3 & 2 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}$.

a. Compute A^{-1}. Check it.

$$\left(\begin{array}{ccc|ccc} 3 & 2 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \begin{array}{c} R_2 \\ R_1 \\ \end{array} \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 1 & 0 \\ 3 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \begin{array}{c} \\ R_2 - 3R_1 \\ \end{array}$$

$$\Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 2 & 3 & 1 & -3 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \begin{array}{c} \\ R_3 \\ R_2 \end{array} \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & 2 & 3 & 1 & -3 & 0 \end{array}\right) \begin{array}{c} \\ \\ R_3 - 2R_2 \end{array}$$

$$\Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & -3 & -2 \end{array}\right) \begin{array}{c} R_1 - R_3 \\ R_2 + 2R3 \\ -R_3 \end{array} \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 4 & 2 \\ 0 & 1 & 0 & 2 & -6 & -3 \\ 0 & 0 & 1 & -1 & 3 & 2 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} -1 & 4 & 2 \\ 2 & -6 & -3 \\ -1 & 3 & 2 \end{pmatrix}$$

Check:

$$\begin{pmatrix} 3 & 2 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} -1 & 4 & 2 \\ 2 & -6 & -3 \\ -1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

b. Solve the equations

$$\begin{aligned} 3x + 2y &= 2 \\ x - z &= 1 \\ y + 2z &= 3 \end{aligned}$$

$AX = B$ with $A = \begin{pmatrix} 3 & 2 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}$ $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $B = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$

$$\Rightarrow X = A^{-1}B \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 & 4 & 2 \\ 2 & -6 & -3 \\ -1 & 3 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 \\ -11 \\ 7 \end{pmatrix}$$

6. (20 points) (Multiple Choice: Circle one) If $C = AB$, then $(C^{\mathsf{T}})^{-1} =$

a. $A^{\mathsf{T}}B^{-1} + A^{-1}B^{\mathsf{T}}$

b. $B^{-1}A^{\mathsf{T}} + B^{\mathsf{T}}A^{-1}$

c. $(A^{-1})^{\mathsf{T}}(B^{-1})^{\mathsf{T}}$ correctchoice

d. $(B^{\mathsf{T}})^{-1}(A^{\mathsf{T}})^{-1}$

Now prove it. You may use any result proved in class or in the book or on homework.

METHOD 1:

	$C = AB$	given
	$C^{\mathsf{T}} = (AB)^{\mathsf{T}} = B^{\mathsf{T}}A^{\mathsf{T}}$	since $(XY)^{\mathsf{T}} = Y^{\mathsf{T}}X^{\mathsf{T}}$
	$(C^{\mathsf{T}})^{-1} = (B^{\mathsf{T}}A^{\mathsf{T}})^{-1} = (A^{\mathsf{T}})^{-1}(B^{\mathsf{T}})^{-1}$	since $(XY)^{-1} = Y^{-1}X^{-1}$
	$(C^{\mathsf{T}})^{-1} = (A^{-1})^{\mathsf{T}}(B^{-1})^{\mathsf{T}}$	since $(X^{\mathsf{T}})^{-1} = (X^{-1})^{\mathsf{T}}$

METHOD 2:

	$C^{\mathsf{T}}\left[(A^{-1})^{\mathsf{T}}(B^{-1})^{\mathsf{T}}\right]$	
	$= (AB)^{\mathsf{T}}\left[(A^{-1})^{\mathsf{T}}(B^{-1})^{\mathsf{T}}\right]$	since $C = AB$
	$= (B^{\mathsf{T}}A^{\mathsf{T}})\left[(A^{-1})^{\mathsf{T}}(B^{-1})^{\mathsf{T}}\right]$	since $(XY)^{\mathsf{T}} = Y^{\mathsf{T}}X^{\mathsf{T}}$
	$= \left\{B^{\mathsf{T}}\left[A^{\mathsf{T}}(A^{-1})^{\mathsf{T}}\right]\right\}(B^{-1})^{\mathsf{T}}$	since associative
	$= \{B^{\mathsf{T}}[\mathbf{1}]\}(B^{-1})^{\mathsf{T}}$	since $(X^{\mathsf{T}})^{-1} = (X^{-1})^{\mathsf{T}}$
	$= B^{\mathsf{T}}(B^{-1})^{\mathsf{T}}$	since identity
	$= \mathbf{1}$	since inverse

So $(C^{\mathsf{T}})^{-1} = (A^{-1})^{\mathsf{T}}(B^{-1})^{\mathsf{T}}$