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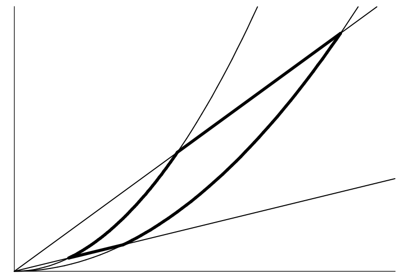
MATH 311 Exam 3 Spring 2003
Section 200 Solutions P. Yasskin

**1.** (15 points) Consider the "diamond shaped" region $D$ bounded by the lines $y = x$ and $y = 3x$ and the parabolas $y = x^2$ and $y = 2x^2$. Compute $\iint_D \frac{y}{x}\, dA$ over the diamond.



Here are some steps to follow:

- Let $u = \frac{y}{x}$ and $v = \frac{y}{x^2}$. Solve for $x$ and $y$.

$\frac{u}{v} = \frac{y}{x}\frac{x^2}{y} = x$ $\qquad \frac{u^2}{v} = \left(\frac{y}{x}\right)^2 \frac{x^2}{y} = y$ $\qquad$ So $x = \frac{u}{v}$ and $y = \frac{u^2}{v}$.

- Find the Jacobian factor.

$$J = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\det\begin{pmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ \frac{2u}{v} & -\frac{u^2}{v^2} \end{pmatrix}\right| = \left|-\frac{u^2}{v^3} - -\frac{2u^2}{v^3}\right| = \frac{u^2}{v^3}$$

- Express the integrand in terms of $u$ and $v$.

$$\frac{y}{x} = \frac{u^2/v}{u/v} = u$$

- Express the boundary curves in terms of $u$ and $v$.

$y = x$ and $y = 3x$ become $u = 1$ and $u = 3$.
$y = x^2$ and $y = 2x^2$ become $v = 1$ and $v = 2$.

- Compute $\iint_D \frac{y}{x}\, dA$.

$$\iint_D \frac{y}{x}\, dA = \int_1^2 \int_1^3 u \frac{u^2}{v^3}\, du\, dv = \int_1^2 v^{-3}\, dv \int_1^3 u^3\, du = \left[\frac{v^{-2}}{-2}\right]_1^2 \left[\frac{u^4}{4}\right]_1^3$$

$$= \left[-\frac{1}{8} + \frac{1}{2}\right]\left[\frac{81}{4} - \frac{1}{4}\right] = \frac{3}{8}20 = \frac{15}{2}$$

**2.** (25 points) Consider the parametric curve $r(t) = \left(2t, t^2, \frac{1}{3}t^3\right)$ for $0 \le t \le 2$.

**a.** (15 pts) Compute $\int_{(0,0,0)}^{(4,4,8/3)} (xy + 3z)\,ds$ along this curve

$\vec{v} = (2, 2t, t^2) \qquad |\vec{v}| = \sqrt{4 + 4t^2 + t^4} = \sqrt{(2+t^2)^2} = 2 + t^2$

$xy + 3z = 2t^3 + t^3 = 3t^3$

$$\int_{(0,0,0)}^{(4,4,8/3)} (xy + 3z)\,ds = \int_0^2 3t^3(2 + t^2)\,dt = \int_0^2 (6t^3 + 3t^5)\,dt = \left[\frac{6t^4}{4} + \frac{3t^6}{6}\right]_0^2 = 24 + 32 = 56$$

**b.** (10 pts) Compute $\int_{(0,0,0)}^{(4,4,8/3)} \vec{F} \cdot d\vec{s}$ along this curve where $\vec{F} = (3z, 2y, x)$.

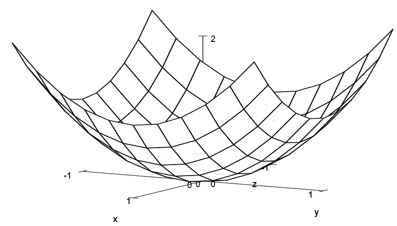
$\vec{v} = (2, 2t, t^2) \qquad \vec{F} = (3z, 2y, x) = (t^3, 2t^2, 2t)$

$$\int_{(0,0,0)}^{(4,4,8/3)} \vec{F} \cdot d\vec{s} = \int_0^2 \vec{F} \cdot \vec{v}\,dt = \int_0^2 (2t^3 + 4t^3 + 2t^3)\,dt = \int_0^2 8t^3\,dt = 2t^4\Big|_0^2 = 32$$

**3.** (30 points) Consider the parametric surface

$$\vec{R}(p,q) = (p, q, p^2 + q^2)$$

for $-1 \le p \le 1$ and $-1 \le q \le 1$.



**a.** (15 pts) Find the total mass $M = \iint \delta\,dS$ on this surface if the surface density is $\delta = \sqrt{4z+1}$.

$$\begin{matrix} & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_p = & (1 & 0 & 2p) \\ \vec{e}_q = & (0 & 1 & 2q) \end{matrix}$$

$\vec{N} = \vec{e}_p \times \vec{e}_q = (-2p, -2q, 1) \qquad \left|\vec{N}\right| = \sqrt{4p^2 + 4q^2 + 1}$

On the surface $\delta = \sqrt{4(p^2 + q^2) + 1}$

$$M = \iint \delta\,dS = \int\int \delta\left|\vec{N}\right| dp\,dq = \int_{-1}^1 \int_{-1}^1 (4p^2 + 4q^2 + 1)\,dp\,dq = \int_{-1}^1 \left[\frac{4p^3}{3} + 4q^2p + p\right]_{p=-1}^1 dq$$

$$= \int_{-1}^1 \left[\frac{8}{3} + 8q^2 + 2\right]_{-1}^1 dq = \left[\frac{8q}{3} + \frac{8q^3}{3} + 2q\right]_{q=-1}^1 = \left[\frac{16}{3} + \frac{16}{3} + 4\right] = \frac{44}{3}$$

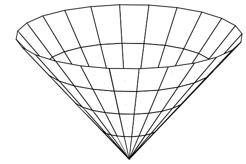
**b.** (15 pts) Find the flux $\iint \vec{F} \cdot d\vec{S}$ of the vector field $\vec{F} = (3x, 3y, 3z)$ through this surface with normal pointing down.

Reverse the normal: $\vec{N} = (2p, 2q, -1) \qquad \vec{F}\left(\vec{R}(p,q)\right) = (3p, 3q, 3(p^2 + q^2))$

$$\iint \vec{F} \cdot d\vec{S} = \iint \vec{F}\left(\vec{R}(p,q)\right) \cdot \vec{N}\,dp\,dq = \int_{-1}^1 \int_{-1}^1 6p^2 + 6q^2 - 3(p^2 + q^2)\,dp\,dq$$

$$= \int_{-1}^1 \int_{-1}^1 3p^2 + 3q^2\,dp\,dq = \int_{-1}^1 \int_{-1}^1 3p^2\,dp\,dq + \int_{-1}^1 \int_{-1}^1 3q^2\,dp\,dq = [q]_{-1}^1[p^3]_{-1}^1 + [p]_{-1}^1[q^3]_{-1}^1 = 8$$

**4.** (30 points) Use 2 methods to compute $\iint_C \vec{F} \cdot d\vec{S}$ for $\vec{F} = (5xz, 5yz, z^2)$



over the conical surface $C$ given by $z = \sqrt{x^2 + y^2} \le 3$
with normal pointing down and out.

**a.** (15 pts) METHOD 1: Compute $\iint_C \vec{F} \cdot d\vec{S}$ directly as a surface integral using the parametrization $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, r)$.

HINT: Find $\vec{e}_r$, $\vec{e}_\theta$, $\vec{N}$ and $\vec{F}$ on the cone.

$$\begin{array}{rcccc} & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_r = & (\ \cos\theta & \sin\theta & 1) \\ \vec{e}_\theta = & (-r\sin\theta & r\cos\theta & 0) \end{array}$$

$\vec{N} = \vec{e}_\varphi \times \vec{e}_\theta = (-r\cos\theta, -r\sin\theta, r)$ $\quad\vec{N}$ is up and in.

Reverse it: $\vec{N} = (r\cos\theta, r\sin\theta, -r)$

On the cone, $\vec{F} = (5xz, 5yz, z^2) = (5r^2\cos\theta, 5r^2\sin\theta, r^2)$

$$\iint_C \vec{F} \cdot d\vec{S} = \int_0^{2\pi}\int_0^3 \vec{F} \cdot \vec{N}\,dr\,d\theta = \int_0^{2\pi}\int_0^3 (5r^3\cos^2\theta + 5r^3\sin^2\theta - r^3)\,dr\,d\theta = \int_0^{2\pi}\int_0^3 4r^3\,dr\,d\theta$$

$$= 2\pi r^4\big|_0^3 = 162\pi$$

**b.** (15 pts) METHOD 2: Compute $\iint_C \vec{F} \cdot d\vec{S}$ by applying Gauss' Theorem

$\iiint_V \vec{\nabla} \cdot \vec{F}\,dV = \iint_{\partial V} \vec{F} \cdot d\vec{S}$ to the solid cone $V$ whose boundary is $\partial V = C + D$

where $C$ is the conical surface and $D$ is the disk at the top of the cone.

For the volume use cylindrical coordinates. $dV = r\,dr\,d\theta\,dz$ $\quad\vec{\nabla} \cdot \vec{F} = 5z + 5z + 2z = 12z$

$$\iiint_V \vec{\nabla} \cdot \vec{F}\,dV = \int_0^{2\pi}\int_0^3\int_r^3 12z\,r\,dz\,dr\,d\theta = 2\pi\int_0^3 [6z^2]_r^3\,r\,dr = 2\pi\int_0^3 (54 - 6r^2)\,r\,dr$$

$$= 2\pi\int_0^3 (54r - 6r^3)\,dr = 2\pi\left[27r^2 - 6\frac{r^4}{4}\right]_0^3 = 2\pi\left(3^5 - \frac{3^5}{2}\right) = 3^5\pi = 243\pi$$

Parametrize the disk as $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, 3)$.

$$\begin{array}{rcccc} & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_r = & (\ \cos\theta & \sin\theta & 0) \\ \vec{e}_\theta = & (-r\sin\theta & r\cos\theta & 0) \end{array}$$

$\vec{N} = \vec{e}_\varphi \times \vec{e}_\theta = (0, 0, r)$ $\quad\vec{N}$ is up as required.

On the disk, $\vec{F} = (5xz, 5yz, z^2) = (15r\cos\theta, 15r\sin\theta, 9)$

$$\iint_D \vec{F} \cdot d\vec{S} = \int_0^{2\pi}\int_0^3 \vec{F} \cdot \vec{N}\,dr\,d\theta = \int_0^{2\pi}\int_0^3 9r\,dr\,d\theta = 2\pi\frac{9r^2}{2}\bigg|_0^3 = 81\pi$$

Apply Gauss' Theorem: $\iint_C \vec{F} \cdot d\vec{S} = \iiint_V \vec{\nabla} \cdot \vec{F}\,dV - \iint_D \vec{F} \cdot d\vec{S} = 243\pi - 81\pi = 162\pi$