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| MATH 311 | Final Exam | Spring 2003 |
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| Section 200 | Solutions | P. Yasskin |

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| 1 | /25 | 3 | /25 |
| 2 | /25 | 4 | /25 |

Work Out: (25 points each)

**1.** Consider an ideal gas whose density, $\rho$, temperature, $T$, and pressure, $P$, are functions of position. Thus if we consider a two dimensional space $\mathbb{R}^2$ whose coordinates are $(\rho,T)$ then the ideal gas law, $P = k\rho T$, defines a function $P : \mathbb{R}^2 \to \mathbb{R}$. (Here $k$ is a constant which may appear in your answers.) Further, the formulas which give $(\rho,T)$ as functions of position $(x,y,z)$ define a function $F : \mathbb{R}^3 \to \mathbb{R}^2$. The composition $P \circ F : \mathbb{R}^3 \to \mathbb{R}$ then gives $P$ as a function of position. At the point $X = (2,3,4)$, $\rho$, $T$ and their gradients are

$$\rho(X) = 2 \qquad T(X) = 78 \qquad \vec{\nabla}\rho(X) = (0.1, 0.2, -0.1) \qquad \vec{\nabla}T(X) = (0.2, -0.3, 0.4)$$

**a.** What is $JF(X) = \dfrac{d(\rho,T)}{d(x,y,z)}(X)$, the Jacobian matrix of $F$ at $X$?

**b.** What are $JP$ and $JP(\rho(X),T(X))$, the Jacobian matrix of $P$ and the Jacobian matrix of $P$ at $X$?

**c.** What is $J(P \circ F)(X)$, the Jacobian matrix of $P \circ F$ at $X$?

**d.** Use the linear approximation to estimate $P(Y)$, the pressure at the point $Y = (2.2, 2.9, 4.1)$.

**e.** At the time $t = 0$, you are at $X = (2, 3, 4)$ and moving with velocity $\vec{v} = (-1, 1, 2)$. Use the linear approximation to estimate the temperature $T$ at time $t = 2$.

The remainder of the exam is customized for each student.

**Exam 1 #2:** Find the non-parametric equation for the plane tangent to the surface $x^3y^2 + xz^3 = 31$ at the point $(x, y, z) = (1, 2, 3)$.

**Exam 1 #4:** Let $M = \begin{pmatrix} 2 & 5 & 4 & -1 \\ 0 & 1 & -2 & 1 \\ 1 & 3 & 0 & -2 \\ 2 & 6 & 3 & x \end{pmatrix}$

**a.** Compute $\det M$ (as a function of $x$).

**b.** For what value(s) of $x$ does $M^{-1}$ exist? Why?

**Exam 1 #5:** Let $A = \begin{pmatrix} 3 & 2 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}$.

**a.** Compute $A^{-1}$. Check it.

**b.** Solve the equations

$$\begin{aligned} 3x + 2y &= 2 \\ x - z &= 1 \\ y + 2z &= 3 \end{aligned}$$

**Exam 1 #6:** (Multiple Choice: Circle one) If $C = AB$, then $(C^{\mathsf{T}})^{-1} =$

**a.** $A^{\mathsf{T}}B^{-1} + A^{-1}B^{\mathsf{T}}$

**b.** $B^{-1}A^{\mathsf{T}} + B^{\mathsf{T}}A^{-1}$

**c.** $(A^{-1})^{\mathsf{T}}(B^{-1})^{\mathsf{T}}$

**d.** $(B^{\mathsf{T}})^{-1}(A^{\mathsf{T}})^{-1}$

Now prove it. You may use any result proved in class or in the book or on homework.

**Exam 2 #1:** Let $(P_2)^2$ be the vector space of ordered pairs of polynomials of degree less than 2. For example,

$$\vec{q} = \begin{pmatrix} 2x-3 \\ 3x+1 \end{pmatrix} \in (P_2)^2 \qquad \text{and} \qquad \vec{q}(2) = \begin{pmatrix} 1 \\ 7 \end{pmatrix}$$

The standard basis of $(P_2)^2$ is

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} x \\ 0 \end{pmatrix} \quad e_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad e_4 = \begin{pmatrix} 0 \\ x \end{pmatrix}$$

Another basis for $(P_2)^2$ is

$$E_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad E_2 = \begin{pmatrix} 1+x \\ 0 \end{pmatrix} \quad E_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad E_4 = \begin{pmatrix} 0 \\ 1+x \end{pmatrix}$$

**a.** Find the change of basis matrices $\underset{E \leftarrow e}{C}$ and $\underset{e \leftarrow E}{C}$.

**b.** Find $(\vec{q})_e$ the components of $\vec{q} = \begin{pmatrix} 2x-3 \\ 3x+1 \end{pmatrix}$ relative to the $e$-basis.

**c.** Find $(\vec{q})_E$ the components of $\vec{q}$ relative to the $E$-basis by using the change of basis matrix.

**d.** If $(\vec{r})_E = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$, what is $\vec{r}$?

**Exam 2 #2:** Let $(P_2)^2$ be the vector space of ordered pairs of polynomials of degree less than $2$. For example,

$$\vec{q} = \begin{pmatrix} 2x-3 \\ 3x+1 \end{pmatrix} \in (P_2)^2 \qquad \text{and} \qquad \vec{q}(2) = \begin{pmatrix} 1 \\ 7 \end{pmatrix}$$

Consider the subspace $S$ of $(P_3)^2$ spanned by $\begin{pmatrix} 1+x \\ 1-x \end{pmatrix}$, $\begin{pmatrix} 2+x \\ 2-x \end{pmatrix}$, $\begin{pmatrix} 3+x \\ 3-x \end{pmatrix}$, $\begin{pmatrix} 1-x \\ 1+x \end{pmatrix}$. Pare the spanning set down to a basis for $S$ and find the dimension of $S$.

**Exam 2 #3:** Let $(P_2)^2$ be the vector space of ordered pairs of polynomials of degree less than 2. For example,

$$\vec{q} = \begin{pmatrix} 2x-3 \\ 3x+1 \end{pmatrix} \in (P_2)^2 \qquad \text{and} \qquad \vec{q}(2) = \begin{pmatrix} 1 \\ 7 \end{pmatrix}$$

The standard basis of $(P_2)^2$ is

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} x \\ 0 \end{pmatrix} \quad e_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad e_4 = \begin{pmatrix} 0 \\ x \end{pmatrix}$$

Another basis for $(P_2)^2$ is

$$E_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad E_2 = \begin{pmatrix} 1+x \\ 0 \end{pmatrix} \quad E_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad E_4 = \begin{pmatrix} 0 \\ 1+x \end{pmatrix}$$

The change of basis matrices are

$$\underset{e\leftarrow E}{C} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \underset{E\leftarrow e}{C} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Now consider the linear map $L : (P_2)^2 \rightarrow P_2$ given by $L(\vec{p}) = p_1 + p_2$. (Just add the two component polynomials.) For example, if $\vec{q} = \begin{pmatrix} -3+2x \\ 1+3x \end{pmatrix}$ then

$$L(\vec{q}) = L\begin{pmatrix} -3+2x \\ 1+3x \end{pmatrix} = (-3+2x) + (1+3x) = -2+5x$$

**a.** Find the matrix of $L$ relative to the $e$-basis on $(P_2)^2$ and the $f$-basis on $P_2$ where $f_1 = 1$ and $f_2 = x$. Call it $\underset{f\leftarrow e}{A}$.

**b.** Find the matrix of $L$ relative to the $E$-basis on $(P_2)^2$ and the $f$-basis on $P_2$ by using the change of basis matrix. Call it $\underset{f\leftarrow E}{B}$.

**c.** Find the matrix of $L$ relative to the $E$-basis on $(P_2)^2$ and the $f$-basis on $P_2$ from the definition.

**d.** If $(\vec{r})_E = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$, what are $[L(\vec{r})]_f$ and $L(\vec{r})$ ?

**Exam 2 #4:** Let $(P_2)^2$ be the vector space of ordered pairs of polynomials of degree less than $2$. For example,

$$\vec{q} = \begin{pmatrix} 2x-3 \\ 3x+1 \end{pmatrix} \in (P_2)^2 \qquad \text{and} \qquad \vec{q}(2) = \begin{pmatrix} 1 \\ 7 \end{pmatrix}$$

Consider the linear map $L : (P_2)^2 \to P_2$ given by $L(\vec{p}) = p_1 + p_2$. When necessary, let $\vec{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} a+bx \\ c+dx \end{pmatrix}$.

**a.** Find the kernel of $L$. Give a basis and the dimension.

**b.** Find the image of $L$. Give a basis and the dimension.

**c.** Is $L$ one-to-one? Why?

**d.** Is $L$ onto? Why?

**e.** Check that the Nullity-Rank Theorem is satisfied.

**Exam 2 #5:** Let $(P_2)^2$ be the vector space of ordered pairs of polynomials of degree less than 2. For example,

$$\vec{q} = \begin{pmatrix} 2x-3 \\ 3x+1 \end{pmatrix} \in (P_2)^2 \qquad \text{and} \qquad \vec{q}(2) = \begin{pmatrix} 1 \\ 7 \end{pmatrix}$$

Verify that the following function is an inner product on $(P_2)^2$ :

$$\langle\ ,\ \rangle : (P_2)^2 \times (P_2)^2 \longrightarrow \mathbb{R} \quad \text{given by} \quad \langle \vec{p}, \vec{q} \rangle = \int_{-1}^{1} p_1(x)q_1(x) + p_2(x)q_2(x)\,dx$$

For example, $\left\langle \begin{pmatrix} 1+x \\ 2x \end{pmatrix}, \begin{pmatrix} -x \\ 2-x \end{pmatrix} \right\rangle = \int_{-1}^{1} (1+x)(-x) + (2x)(2-x)\,dx = \int_{-1}^{1} (3x - 3x^2)\,dx = -2$

**a.** Symmetric:

**b.** Bilinear:

**c.** Positive Definite:

**Exam 2 #6:** Let $(P_2)^2$ be the vector space of ordered pairs of polynomials of degree less than $2$. For example,

$$\vec{q} = \begin{pmatrix} 2x-3 \\ 3x+1 \end{pmatrix} \in (P_2)^2 \qquad \text{and} \qquad \vec{q}(2) = \begin{pmatrix} 1 \\ 7 \end{pmatrix}$$

Using the following inner product on $(P_2)^2$ :

$$\langle \ , \ \rangle : (P_2)^2 \times (P_2)^2 \to \mathbb{R} \quad \text{given by} \quad \langle \vec{p}, \vec{q} \rangle = \int_{-1}^{1} p_1(x)q_1(x) + p_2(x)q_2(x)\,dx$$

find the angle between the vectors $\begin{pmatrix} 1 \\ x \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -x \end{pmatrix}$.

**Exam 3 #2:** Consider the parametric curve $r(t) = \left(2t, t^2, \frac{1}{3}t^3\right)$ for $0 \le t \le 2$.

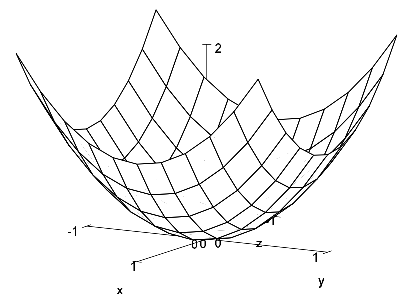
**a.** Compute $\int_{(0,0,0)}^{(4,4,8/3)} (xy + 3z)\, ds$ along this curve

**b.** Compute $\int_{(0,0,0)}^{(4,4,8/3)} \vec{F} \cdot d\vec{s}$ along this curve where $\vec{F} = (3z, 2y, x)$.

**Exam 3 #3:** Consider the parametric surface

$$\vec{R}(p,q) = (p, q, p^2 + q^2)$$

for $-1 \le p \le 1$ and $-1 \le q \le 1$.

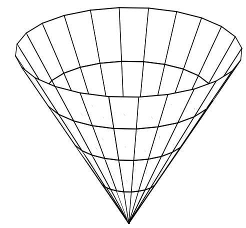



**a.** Find the total mass $M = \iint \delta \, dS$ on this surface if the surface density is $\delta = \sqrt{4z+1}$.

**b.** Find the flux $\iint \vec{F} \cdot d\vec{S}$ of the vector field $\vec{F} = (3x, 3y, 3z)$ through this surface with normal pointing down.

**Exam 3 #4:** Use 2 methods to compute

$$\iint_C \vec{F} \cdot d\vec{S} \quad \text{for} \quad \vec{F} = (5xz, 5yz, z^2)$$



over the conical surface $C$ given by

$$z = \sqrt{x^2 + y^2} \leq 3$$

with normal pointing down and out.

**a.** METHOD 1: Compute $\iint_C \vec{F} \cdot d\vec{S}$ directly as a surface integral using the parametrization $\vec{R}(r, \theta) = (r\cos\theta, r\sin\theta, r)$.

HINT: Find $\vec{e}_r$, $\vec{e}_\theta$, $\vec{N}$ and $\vec{F}$ on the cone.

Recall: $\vec{F} = (5xz, 5yz, z^2)$ and $C$ is the conical surface $z = \sqrt{x^2 + y^2} \le 3$ with normal pointing down and out.

**b.** METHOD 2: Compute $\iint\limits_{C} \vec{F} \cdot d\vec{S}$ by applying Gauss' Theorem

$\iiint\limits_{V} \vec{\nabla} \cdot \vec{F}\, dV = \iint\limits_{\partial V} \vec{F} \cdot d\vec{S}$ to the solid cone $V$ whose boundary is $\partial V = C + D$ where $C$ is the conical surface and $D$ is the disk at the top of the cone.