In most of this exam, we will consider three 3-dimensional manifolds with different metrics.

- **Manifold (a)** is ordinary flat 3-space written in (almost) spherical coordinates. So its metric is
$$ds^2 = a^2[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)]$$
where a is a dimensionless constant which simply sets the scale for all distances. (The constant a is the reason I said "almost.")

- **Manifold (b)** is a 3-dimensional manifold with the metric
$$ds^2 = a^2\left[\frac{1}{1-r^2}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\right]$$

- **Manifold (c)** is a 3-dimensional manifold with the metric
$$ds^2 = a^2\left[\frac{1}{1+r^2}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\right]$$

In all 3 cases, the surface $r = \rho$ is a sphere of radius $a\rho$ whose metric is
$$ds^2 = a^2\rho^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

1. Do the 2-dimensional integral which computes the surface area, A, of the sphere $r = \rho$.

2. In all 3 cases, compute the arclength, R, of the radial line from the origin to a point on the sphere with constant $\theta = \theta_o$ and $\varphi = \varphi_o$. This is the parametric curve
$$(r,\theta,\varphi) = \vec{r}(t) = (t,\theta_o,\varphi_o) \qquad \text{for} \quad 0 \le t \le \rho$$
In case (a) the answer should be $R = a\rho$.

3. The equator of the sphere is the circle with $\theta = \frac{\pi}{2}$. Find the circumference, C, of the equator.

4. In each of the 3 cases, what is the ratio of the circumference, C, to the radial arclength, R? Is this bigger or smaller than 2π?

5. In each of the 3 cases, is the area, A, of the sphere bigger or smaller than $4\pi R^2$?

6. In all 3 cases, do the 3-dimensional integral which computes the volume of the ball enclosed in the sphere $r = \rho$. Is this bigger or smaller than $\frac{4}{3}\pi R^3$?

7. In all 3 cases, compute the Christoffel symbols, the Riemann curvature, the Ricci curvature and the scalar curvature. You must compute this by hand, but you can use Maple or Mathematica to check your work. If you do, also hand in the Maple or Mathematica worksheet you produced. I suggest you do all 3 computations at the same time by writing the metric as
$$ds^2 = a^2\left[\frac{1}{1-pr^2}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\right]$$
where
$$p = \begin{cases} 0 & \text{for case (a)} \\ 1 & \text{for case (b)} \\ -1 & \text{for case (c)} \end{cases}$$
In that case, you only need to substitute the values of p into the answers for the mixed index (one up, one down) Ricci and scalar curvatures.

Note: There are 10 non-zero components of Christoffel, 12 non-zero components of Riemann and 3 non-zero components of Ricci.

8. In cases (b) and (c), find a coordinate transformation $r = f(\check{r})$ so the metric can be written as

$$ds^2 = a^2\left[d\check{r}^2 + g(\check{r})^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\right]$$

Note: In case (a) the metric is already in this form with $r = \check{r}$ and $g(\check{r}) = \check{r}$.
In cases (b) and (c), what are $f(\check{r})$ and $g(\check{r})$?
Note: A boundary condition on $f(\check{r})$ is $f(0) = 0$.

9. In case (b), notice that when $r = 1$ something goes wrong with the metric: g_{rr} becomes infinite. Further, if $r > 1$ then the signature of the metric changes: $g_{rr} < 0$. Is $r = 1$ a singularity in the manifold or a coordinate singularity? If you look at the transformation to the $\check{r}$ coordinate, i.e. $r = f(\check{r})$, you will see that the interval $0 \le r \le 1$ corresponds to the interval $0 \le \check{r} \le \frac{\pi}{2}$. Further, if you look at the metric in the $\check{r}$ coordinate, you will see that the metric is perfectly normal at $\check{r} = \frac{\pi}{2}$; i.e. there is no 0 or ∞. Further there is no problem going beyond $\check{r} = \frac{\pi}{2}$; the metric stays positive definite. In addition, the metric is invariant under the transformation $\check{r} \to \check{r}' = \pi - \check{r}$. So the point with $\check{r} = \pi$ is no different than the point $\check{r} = 0$. It is a second "origin" of the coordinate system, like the north and south poles on a sphere. In fact manifold (b) is a 3-sphere and $\check{r} = \frac{\pi}{2}$ is the equator. (And manifold (c) is a 3-hypersphere.) Compute the volume of the whole 3-sphere in two ways: first using the r coordinate to get the volume of a hemisphere, which is a special case of problem 6, second using the $\check{r}$ coordinate.

We now turn to stationary, homogeneous, isotropic, spacetimes. The 3 cases are:

Case (a):

$$ds^2 = -dt^2 + a(t)^2[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)]$$

Case (b):

$$ds^2 = -dt^2 + a(t)^2\left[\frac{1}{1-r^2}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\right]$$

Case (c):

$$ds^2 = -dt^2 + a(t)^2\left[\frac{1}{1+r^2}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\right]$$

10. In all 3 cases, use Maple or Mathematica to compute the Christoffel symbols, the Riemann curvature, the Ricci curvature, the scalar curvature and the Einstein curvature. Once again, I suggest you do all 3 computations at the same time by writing the metric as

$$ds^2 = -dt^2 + a(t)^2\left[\frac{1}{1-pr^2}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\right]$$

where

$$p = \begin{cases} 0 & \text{for case (a)} \\ 1 & \text{for case (b)} \\ -1 & \text{for case (c)} \end{cases}$$

In that case, you only need to substitute the values of p into the answers for the Einstein curvatures.