Consider the elliptic coordinate system:

$$x = 4t\cos\varphi$$
$$y = 3t\sin\varphi$$

The t-curve for $\varphi = \varphi_0$ is the radial ray

$$\frac{y}{x} = \frac{3}{4}\tan\varphi_0$$

which may be parametrized by

$$\vec{r}_1(t) = \begin{pmatrix} 4t\cos\varphi_0 \\ 3t\sin\varphi_0 \end{pmatrix}$$

The φ-curve for $t = t_0$ is the ellipse

$$\frac{x^2}{16} + \frac{y^2}{9} = {t_0}^2$$

which may be parametrized by

$$\vec{r}_2(\varphi) = \begin{pmatrix} 4t_0\cos\varphi \\ 3t_0\sin\varphi \end{pmatrix}$$

We take the xy-coordinate tangent basis vectors to be

$$\hat{\imath}_1 = \hat{\imath}_x = \hat{\imath} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\hat{\imath}_2 = \hat{\imath}_y = \hat{\jmath} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

So the xy-basis is $\hat{\imath} = (\hat{\imath}_1, \hat{\imath}_2)$ and any vector $\vec{v} = \begin{pmatrix} v^1 \\ v^2 \end{pmatrix}$ can be written as

$$\vec{v} = \hat{\imath}\vec{v}_i = \hat{\imath}_1 v^1 + \hat{\imath}_2 v^2 = \hat{\imath} v^1 + \hat{\jmath} v^2$$

where the components of $\vec{v}$ relative to the $\hat{\imath}$ basis are $\vec{v}_i = \begin{pmatrix} v^1 \\ v^2 \end{pmatrix}$ which since we are in the standard basis are exactly the same as $\vec{v}$ itself.

1) Find the $t\varphi$-coordinate tangent basis vectors

$$\vec{e}_1 = \vec{e}_t = \frac{d}{dt}\vec{r}_1(t)$$
$$\vec{e}_2 = \vec{e}_\varphi = \frac{d}{dt}\vec{r}_2(\varphi)$$

So the $t\varphi$-basis is $\vec{e} = (\vec{e}_1, \vec{e}_2)$ and any vector $\vec{v} = \begin{pmatrix} v^1 \\ v^2 \end{pmatrix}$ can be written as

$$\vec{v} = \vec{e}\vec{v}_e = \vec{e}_1 v_e^1 + \vec{e}_2 v_e^2$$

where the components of $\vec{v}$ relative to the $\vec{e}$ basis are $\vec{v}_e = \begin{pmatrix} v_e^1 \\ v_e^2 \end{pmatrix}$.

2) Find the change of basis matrix, $\underset{i\leftarrow e}{C}$, from the $\vec{e}$-basis to the $\hat{\imath}$-basis.

3) Find the change of basis matrix, $\underset{e\leftarrow i}{C}$, from the $\hat{\imath}$-basis to the $\vec{e}$-basis.

4) Given that the $\hat{\imath}$-components of $\vec{v}$ are $\vec{v}_i = \begin{pmatrix} v^1 \\ v^2 \end{pmatrix}$, find the $\vec{e}$-components $\vec{v}_e$.

5) Verify that you have the correct $\vec{v}_e$ by computing $\vec{e}\,\vec{v}_e$ to see you do in fact get back $\vec{v}$.

Consider the linear map $L : \mathbb{R}^2 \to \mathbb{R}^2$ given (in the standard basis) by

$$L\begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = \begin{pmatrix} 4v^1 - 3v^2 \\ 2v^1 + v^2 \end{pmatrix}$$

6) Find the matrix of L relative to the standard basis $\hat{\imath}$ on both the domain and co-domain. Call it $\underset{i\leftarrow i}{L}$.
Compute it by finding $L(\hat{\imath}_p)$ and expanding in the i-basis.

7) Find the matrix of L relative to the i-basis on the domain and the e-basis on the co-domain. Call it $\underset{e\leftarrow i}{L}$.
Compute it using a change of basis matrix. Verify it by computing finding $L(\hat{\imath}_p)$.

8) Find the matrix of L relative to the e-basis on the domain and the i-basis on the co-domain. Call it $\underset{i\leftarrow e}{L}$.
Compute it using a change of basis matrix. Verify it by computing finding $L(\vec{e}_q)$.

9) Find the matrix of L relative to the $t\varphi$-basis $\vec{e}$ on both the domain and co-domain. Call it $\underset{e\leftarrow e}{L}$.
Compute it using change of basis matrices. Verify it by computing finding $L(\vec{e}_q)$.

10) Convert the vector $\vec{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ into the e-basis using the change of basis matrix. Compute $[L(\vec{v})]_e$, the components of $L(\vec{v})$ relative to the e-basis by using $\underset{e\leftarrow e}{L}$. Use this to compute $L(\vec{v}) = \vec{e}\,[L(\vec{v})]_e$. Verify $L(\vec{v})$ agrees with the original definition of L.

11) Compute the components of the inner product (the standard dot product) relative to the i-basis. (Trivial) Call it g^i_{pq}.

12) Compute the components of the inner product relative to the e-basis by finding all the inner products. Call it g^e_{rs}. Verify this is correct by using change of basis matrices.

13) Convert the vector $\vec{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\vec{u} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ into the e-basis using the change of basis matrix. Compute $\langle \vec{v}, \vec{u} \rangle = \vec{v}_e^T g^e \vec{u}_e$. Verify this is equal to $\vec{v} \cdot \vec{u} = \vec{v}_i^T g^i \vec{u}_i$.

14) Find the inverse of g^i. Call it g_i.

15) Find the inverse of g^e. Call it g_e.

16) Given the vector $\vec{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, find the co-vector $v^\flat = g(v, .)$. Give the components relative to both the i-basis and the e-basis, which are called $v^{\flat i}$ and $v^{\flat e}$.