 Homework 2

Consider the elliptic coordinate system:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \vec{R}(t,\varphi) = \begin{pmatrix} 4t\cos\varphi \\ 3t\sin\varphi \end{pmatrix} \tag{i}$$

This can be inverted to give:

$$\begin{pmatrix} t \\ \varphi \end{pmatrix} = \vec{R}^{-1}(x,y) = \begin{pmatrix} \sqrt{\left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2} \\ \arctan\left(\frac{4y}{3x}\right) + \left\{ \begin{matrix} 0 & \text{in I and IV} \\ \pi & \text{in II and III} \end{matrix} \right\} \end{pmatrix} \tag{ii}$$

The xy-coordinate tangent basis vectors are:

$$\hat{\imath}_x = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \hat{\imath}_y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The $t\varphi$-coordinate tangent basis vectors are:

$$\vec{e}_t = \frac{\partial}{\partial t}\vec{R}(t,\varphi) = \begin{pmatrix} 4t\cos\varphi \\ 3t\sin\varphi \end{pmatrix} \qquad \vec{e}_\varphi = \frac{\partial}{\partial \varphi}\vec{R}(t,\varphi) = \begin{pmatrix} -4t\sin\varphi \\ 3t\cos\varphi \end{pmatrix}$$

So we can write the $t\varphi$-basis vectors as linear combinations of the xy-basis vectors:

$$\begin{aligned} \vec{e}_t &= 4t\cos\varphi\,\hat{\imath}_x + 3t\sin\varphi\,\hat{\imath}_y \\ \vec{e}_\varphi &= -4t\sin\varphi\,\hat{\imath}_x + 3t\cos\varphi\,\hat{\imath}_y \end{aligned} \tag{1}$$

1) Invert (1) to write the xy-basis vectors as linear combinations of the $t\varphi$-basis vectors

$$\begin{aligned} \hat{\imath}_x &= \_\_\_\_\_\_\_\_\,\vec{e}_t + \_\_\_\_\_\_\_\_\,\vec{e}_\varphi \\ \hat{\imath}_y &= \_\_\_\_\_\_\_\_\,\vec{e}_t + \_\_\_\_\_\_\_\_\,\vec{e}_\varphi \end{aligned} \tag{2}$$

Let θ^x and θ^y be the dual basis to $\hat{\imath}_x$ and $\hat{\imath}_y$:

$$\theta^x(\hat{\imath}_x) = 1 \qquad \theta^x(\hat{\imath}_y) = 0 \qquad \theta^y(\hat{\imath}_x) = 0 \qquad \theta^y(\hat{\imath}_y) = 1$$

Let ω^t and ω^φ be the dual basis to $\vec{e}_t$ and $\vec{e}_\varphi$:

$$\omega^t(\vec{e}_t) = 1 \qquad \omega^t(\vec{e}_\varphi) = 0 \qquad \omega^\varphi(\vec{e}_t) = 0 \qquad \omega^\varphi(\vec{e}_\varphi) = 1$$

2) Express ω^t and ω^φ as linear combinations of θ^x and θ^y.

$$\begin{aligned} \omega^t &= \_\_\_\_\_\_\_\_\,\theta^x + \_\_\_\_\_\_\_\_\,\theta^y \\ \omega^\varphi &= \_\_\_\_\_\_\_\_\,\theta^x + \_\_\_\_\_\_\_\_\,\theta^y \end{aligned} \tag{3}$$

3) Express θ^x and θ^y as linear combinations of ω^t and ω^φ.

$$\begin{aligned} \theta^x &= \_\_\_\_\_\_\_\_\,\omega^t + \_\_\_\_\_\_\_\_\,\omega^\varphi \\ \theta^y &= \_\_\_\_\_\_\_\_\,\omega^t + \_\_\_\_\_\_\_\_\,\omega^\varphi \end{aligned} \tag{4}$$

4) Consider a function $f(x,y)$. Use the chain rule and (*) to express $\frac{\partial f}{\partial t}$ and $\frac{\partial f}{\partial \varphi}$ as linear combinations of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. Then drop the f's.

$$\frac{\partial}{\partial t} = \_\_\_\_\_\_\_ \frac{\partial}{\partial x} + \_\_\_\_\_\_\_ \frac{\partial}{\partial y} \qquad (5)$$

$$\frac{\partial}{\partial \varphi} = \_\_\_\_\_\_\_ \frac{\partial}{\partial x} + \_\_\_\_\_\_\_ \frac{\partial}{\partial y}$$

5) Consider a function $g(t,\varphi)$. Use the chain rule and (**) to express $\frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial y}$ as linear combinations of $\frac{\partial g}{\partial t}$ and $\frac{\partial g}{\partial \varphi}$. Express the coefficients as functions of t and φ. Then drop the g's.

$$\frac{\partial}{\partial x} = \_\_\_\_\_\_\_ \frac{\partial}{\partial t} + \_\_\_\_\_\_\_ \frac{\partial}{\partial \varphi} \qquad (6)$$

$$\frac{\partial}{\partial y} = \_\_\_\_\_\_\_ \frac{\partial}{\partial t} + \_\_\_\_\_\_\_ \frac{\partial}{\partial \varphi}$$

6) What do you observe about equations (1) and (2) vs. (5) and (6)?

7) Start with equation (*) and express the differentials of x and y as linear combinations of the differentials of t and φ.

$$dx = \_\_\_\_\_\_\_ dt + \_\_\_\_\_\_\_ d\varphi \qquad (7)$$

$$dy = \_\_\_\_\_\_\_ dt + \_\_\_\_\_\_\_ d\varphi$$

8) Start with equation (**) and express the differentials of t and φ as linear combinations of the differentials of x and y. Express the coefficients as functions of t and φ.

$$dt = \_\_\_\_\_\_\_ dx + \_\_\_\_\_\_\_ dy \qquad (8)$$

$$d\varphi = \_\_\_\_\_\_\_ dx + \_\_\_\_\_\_\_ dy$$

9) What do you observe about equations (3) and (4) vs. (7) and (8)?

10) In any basis, the components of the metric are defined by $g^e_{pq} = \vec{e}_p \cdot \vec{e}_q$. In rectangular coordinates, the metric is $g^i_{pq} = \delta_{pq}$ which says that $\hat{\imath}_x$ and $\hat{\imath}_y$ are perpendicular unit vectors. Find g^e_{pq}, the components of the metric in elliptical coordinates, by taking the dot products of $\vec{e}_t$ and $\vec{e}_\varphi$ as given in (1). Then find the inverse matrix g_e^{pq}.

We now need to define covariant derivatives. For the derivative of a function, f, the covariant derivative is just the directional derivative:

$$\nabla_{\vec{v}} f = \vec{v} \cdot \vec{\nabla} f = \sum_p v^p \frac{\partial f}{\partial x^p} \equiv \sum_p v^p f_{,p}$$

where partial derivatives are denoted by a comma. If the direction is a coordinate basis vectors, i.e. $\vec{v} = \vec{e}_p$, then:

$$\nabla_p f \equiv \nabla_{\vec{e}_p} f = \frac{\partial f}{\partial x^p} \equiv f_{,p}$$

For the derivative of a vector, $\vec{u}$, the covariant derivative is defined by the product rule with the understanding that any derivative of the standard rectangular basis vectors is 0. So in rectangular coordinates, $\vec{u} = \sum_q u_i^q \hat{\imath}_q$:

$$\nabla_{\vec{v}} \vec{u} = \sum_q [(\nabla_{\vec{v}} u_i^q)\hat{\imath}_q + u_i^q(\nabla_{\vec{v}} \hat{\imath}_q)] = \sum_q (\nabla_{\vec{v}} u_i^q)\hat{\imath}_q = \sum_q \sum_p v^p u^q_{i,p} \hat{\imath}_q$$

If the direction is a rectangular basis vector, i.e. $\vec{v} = \hat{\imath}_p$, then:

$$\nabla_p \vec{u} \equiv \nabla_{\hat{\imath}_p} \vec{u} = \sum_q u^q_{i,p} \hat{\imath}_q$$

More generally, in any cordinate basis, $\vec{u} = \sum_p u^q_e \vec{e}_q$

$$\nabla_{\vec{v}} \vec{u} = \sum_q [(\nabla_{\vec{v}} u^q_e)\vec{e}_q + u^q_e(\nabla_{\vec{v}}\vec{e}_q)] = \sum_q \sum_p v^p [(\nabla_p u^q_e)\vec{e}_q + u^q_e(\nabla_p \vec{e}_q)]$$

$$= \sum_q \sum_p v^p \left[u^q_{e,p}\vec{e}_q + u^q_e \sum_n \Gamma^n{}_{qp}\vec{e}_n \right]$$

where the connection coefficients, $\Gamma^i{}_{jk}$, are defined by the equation

$$\nabla_p \vec{e}_q = \sum_n \Gamma^n{}_{qp}\vec{e}_n \qquad \text{(iii)}$$

In the last term we interchange the dummy indices q and n to arrive at:

$$\nabla_{\vec{v}} \vec{u} = \sum_p \sum_q v^p \left[u^q_{e,p}\vec{e}_q + \sum_n u^n_e \Gamma^q{}_{np}\vec{e}_q \right]$$

$$= \sum_p \sum_q v^p \left[u^q_{e,p} + \sum_n \Gamma^q{}_{np} u^n_e \right]\vec{e}_q \equiv \sum_p \sum_q v^p u^q_{e;p}\vec{e}_q$$

where the components of the covariant derivative are denoted by a semicolon.

$$u^q_{e;p} = u^q_{e,p} + \sum_n \Gamma^q{}_{np} u^n_e \qquad \text{(iv)}$$

If the direction is a coordinate basis vector, i.e. $\vec{v} = \vec{e}_p$, then:

$$\nabla_p \vec{u} \equiv \nabla_{\vec{e}_p} \vec{u} = \sum_q u^q_{e;p}\vec{e}_q$$

We now want to find the connection coefficients in elliptical coordinates.

11) Start by applying ∇_t and ∇_φ to each equation in (1) and remember that any derivative of $\hat{\imath}_x$ or $\hat{\imath}_y$is 0. The results for $\nabla_t\vec{e}_t$, $\nabla_t\vec{e}_\varphi$, $\nabla_\varphi\vec{e}_t$ and $\nabla_\varphi\vec{e}_\varphi$ should be linear combinations of $\hat{\imath}_x$ and $\hat{\imath}_y$. Reexpress them as linear combinations of $\vec{e}_t$ and $\vec{e}_\varphi$. Use these to read off the 8 connection coefficients from (***).

12) Let $\vec{R}(r^q) = (x^n(r^q))$ be an arbitrary coordinate system. The coordinate basis vectors are, $\vec{e}_q = \sum_n \frac{\partial x^n}{\partial r^q}\hat{\imath}_n$. Use this infromation to show the connection coefficients are symmetric:

$$\Gamma^n{}_{qp} = \Gamma^n{}_{pq} \qquad \text{(v)}$$

The generalization of (iv) to covariant tensors of rank 2, including the metric tensor, is:

$$g_{pr;n} = g_{pq,n} - \Gamma^m{}_{pn} g_{mq} - \Gamma^m{}_{qn} g_{pm}$$

For the metric tensor in rectangular coordinates, $g_{pq} = \delta_{pq}$, and all the Γ's are 0. So

$$g_{pr;n} = 0,$$

which is true in any basis. In a coordinate basis,

$$g_{pr;n} = g_{pq,n} - \Gamma^m{}_{pn} g_{mq} - \Gamma^m{}_{qn} g_{pm} = 0 \qquad \text{(vi)}$$

13) Use (v) and (vi) to show:

$$\Gamma^m{}_{pq} = \frac{1}{2} g^{mn}_e (g^e_{nq,p} + g^e_{np,q} - g^e_{pq,n}) \qquad \text{(vii)}$$

14) Using g^e_{pq} and g^{mn}_e from #10, recompute the 8 connection coefficients for elliptical coordinates.