

Special Relativity & Electromagnetism

In the following problems, we study electromagnetism in special relativity. We regard spacetime as the vector space $\mathbb{R}^4$ with a Lorentz signature metric (pseudo-inner product). Thus, if we choose the orthonormal basis to be

$$e_0 = (1,0,0,0) \qquad e_1 = (0,1,0,0) \qquad e_2 = (0,0,1,0) \qquad e_3 = (0,0,0,1)$$

(so that all indices run from 0 to 3) and the dual basis to be ω^α, then the metric is

$$\eta = \eta_{\alpha\beta}\omega^\alpha \otimes \omega^\beta \qquad \eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and the inverse metric is

$$\eta^{-1} = \eta^{\alpha\beta} e_\alpha \otimes e_\beta \qquad \eta^{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Derivatives will be denoted by $\partial_\alpha = \frac{\partial}{\partial x^\alpha}$ and indices will be lowered and raised using η and η^{-1}.
Use Greek letters, $\alpha\beta\gamma$, to denote $0,1,2,3$ and use latin leters, ijk, to denote $1,2,3$.

Throughout, we will study the electromagnetic field which is the 2-form

$$F = F_{\alpha\beta}\omega^\alpha \otimes \omega^\beta \qquad F_{\alpha\beta} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix},$$

(Notice that $F_{ij} = \varepsilon_{ijk}B^k$.) the electromagnetic potential which is the 1-form

$$A = A_\alpha\omega^\alpha \qquad A_\alpha = (\phi, A_1, A_2, A_3)$$

and the electromagnetic current which is the vector

$$J = J^\alpha e_\alpha \qquad J^\alpha = (\rho, J^1, J^2, J^3).$$

---

1. Write out the components of F with its indices raised, i.e., $F^{\alpha\beta}$.

2. Write out the components of the Hodge dual of F:

$$*F = \frac{1}{2}\varepsilon(F,\bullet,\bullet)$$

3. Write out the independent non-zero components of the Hodge dual of J:

$$*J = \varepsilon(J,\bullet,\bullet,\bullet)$$

4. Show that the rank 3 tensor

$$S_{\alpha\beta\gamma} = \partial_\gamma F_{\alpha\beta} + \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha}$$

is totally antisymmetric, and hence is a 3-form. (HINT: There are three pairs of indices to transpose.)

5. Write out the components of the equations

$$\partial_\gamma F_{\alpha\beta} + \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} = 0 \qquad \partial_\beta F^{\alpha\beta} = 4\pi J^\alpha$$

to see that these are the Maxwell equations:

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \cdot \vec{E} = 4\pi\rho$$
$$\vec{\nabla} \times \vec{E} + \partial_t \vec{B} = 0 \qquad \vec{\nabla} \times \vec{B} - \partial_t \vec{E} = 4\pi \vec{J}$$

6. Write out the components of the equations

$$\partial_\gamma * F_{\alpha\beta} + \partial_\alpha * F_{\beta\gamma} + \partial_\beta * F_{\gamma\alpha} = 4\pi {}^*J_{\alpha\beta\gamma} \qquad \partial_\beta * F^{\alpha\beta} = 0$$

to see they are also the Maxwell equations.

---

7. Write out the components of the equations

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$$

to find expressions for $\vec{E}$ and $\vec{B}$ in terms of ϕ and $\vec{A}$. (NOTE: ϕ is the negative of the usual scalar potential.)

8. Show (in 4-dimensional notation) that if

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$$

is satisfied, then

$$\partial_\gamma F_{\alpha\beta} + \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} = 0$$

is automatically satisfied (provided mixed partials are equal). (NOTE: The 3-dimensional version of these equations is the pair of identities: $\vec{\nabla} \cdot \vec{\nabla} \times \vec{A} = 0$ and $\vec{\nabla} \times \vec{\nabla}\phi = 0$.) Consequently, this subset of the Maxwell equations is actually an identity, and is also refered to as the electromagnetic Bianchi identity.

9. Substitute

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$$

into the remaining Maxwell equation

$$\partial_\beta F^{\alpha\beta} = 4\pi J^\alpha$$

to obtain the Maxwell equation for A_α.

10. Write out the components of the equation

$$\partial_\alpha J^\alpha = 0$$

to see that this is conservation of electric charge. Show that if

$$\partial_\beta F^{\alpha\beta} = 4\pi J^\alpha$$

is satisfied, then

$$\partial_\alpha J^\alpha = 0$$

is automatically satisfied. This is sometimes referred to as an automatic conservation law.

---

11. Let χ be a function. Also let

$$A'_\alpha = A_\alpha + \partial_\alpha \chi \qquad F'_{\alpha\beta} = \partial_\alpha A'_\beta - \partial_\beta A'_\alpha.$$

This is called a gauge transformation. Relate $F'_{\alpha\beta}$ to $F_{\alpha\beta}$ to see that the electromagnetic field is gauge invariant.

12. Given A_α, show that there always exists a function χ such that A'_α satisfies

$$\partial^\alpha A'_\alpha = 0.$$

(NOTE: You may assume that it is always possible to solve a wave equation with an arbitrary but specified source.) This is called the Lorentz gauge. Write out the Maxwell equations for A'_α in the Lorentz gauge to see that the Maxwell equations may always be solved for any arbitrary but specified current J^α.

---

13. Write out the function

$$\mathrm{L} = -\frac{1}{4} F^{\gamma\delta} F_{\gamma\delta}$$

in terms of $\vec{E}$ and $\vec{B}$. This function is called the Lagrangian density for the vacuum electromagnetic field. It is sometimes interpreted as the difference between the kinetic energy $\frac{1}{2}\left|\vec{E}\right|^2$ and the potential energy $\frac{1}{2}\left|\vec{B}\right|^2$. Also write out the Lagrangian density in terms of ϕ and $\vec{A}$.

14. Write out the components of the tensor

$$T^{\alpha\beta} = \frac{1}{4\pi}\left(F^{\alpha\gamma} F^{\beta}_{\ \gamma} - \frac{1}{4}\eta^{\alpha\beta} F^{\gamma\delta} F_{\gamma\delta}\right)$$

in terms of $\vec{E}$ and $\vec{B}$, to find the electromagnetic energy density, momentum density, energy current and momentum current (or stress). This tensor is called the Maxwell energy-momentum-stress tensor.

15. In 4-dimensional notation, compute the divergence of the energy-momentum tensor and use the Bianchi identities and the Maxwell equations to show that

$$\partial_\beta T^{\alpha\beta} = -J_\beta F^{\alpha\beta}.$$

This shows that the electromagnetic energy-momentum is not conserved if the current is non-zero. The reason for this is that we have ignored the energy-momentum of the charged particles producing the current. In problem 16, we study the motion of a charged particle. Then in problem 17 we study the energy-momentum tensor of a fluid of charged particles.

16. A particle of mass m with electric charge q is moving on the parametrized path $x^\alpha(\tau)$ where τ is the proper time. Consequently, it has unit timelike tangent vector

$$U = U^\alpha e_\alpha \qquad\qquad U^\alpha = \frac{\partial x^\alpha}{\partial \tau} = (\gamma, \gamma v^1, \gamma v^2, \gamma v^3)$$

and where

$$\gamma = \frac{1}{\sqrt{1 - |\vec{v}|^2}}.$$

Further, its 4-momentum is

$$p^\alpha = mU^\alpha.$$

Write out the components of the equations

$$U^\beta \partial_\beta (p^\alpha) = qU^\beta F^\alpha_{\ \beta}$$

to obtain the Lorentz force and power laws. (HINTS: Don't expand p^α . Be careful with the factors of γ.)

17. Consider a fluid of charged particles of rest mass m and charge q, with fluid velocity U^α and energy density ρ in the instantaneous local rest frame. Then the charge density in the instantaneous local rest frame is $\frac{q}{m}\rho$ and the electromagnetic current is

$$J^\alpha = \frac{q}{m}\rho U^\alpha$$

Assuming that the particles are non-interacting except for their electromagnetic forces, then the energy momentum tensor for the fluid is that for dust:

$$T^{\alpha\beta}_{\text{fluid}} = \rho U^\alpha U^\beta$$

each particle moves according to the Lorentz force equation:

$$U^\beta \partial_\beta (mU^\alpha) = qU^\beta F^\alpha_{\ \beta}$$

the energy momentum tensor for the electromagnetic field is

$$T^{\alpha\beta}_{\text{em}} = \frac{1}{4\pi}(F^{\alpha\gamma} F^{\beta}_{\ \gamma} - \frac{1}{4}\eta^{\alpha\beta} F^{\gamma\delta} F_{\gamma\delta}),$$

and the electromagnetic field satisfies the Bianchi identities and the Maxwell equations with current J^α. Then, as seen in problem 10, the electromagnetic current is conserved:

$$\partial_\alpha J^\alpha = 0$$

and, as seen in problem 15, the electromagnetic energy-momentum tensor satisfies:

$$\partial_\beta T^{\alpha\beta}_{\text{em}} = -J_\beta F^{\alpha\beta}.$$

Now use the Lorentz force equation and the conservation of electromagnetic current to show that the fluid energy-momentum tensor satisfies:

$$\partial_\beta T^{\alpha\beta}_{\text{fluid}} = J_\beta F^{\alpha\beta}.$$

(HINT: Factor $T^{\alpha\beta}_{\text{fluid}}$ as

$$T^{\alpha\beta}_{\text{fluid}} = U^\alpha(\rho U^\beta)$$

and use the product rule.) Thus the total energy-momentum is conserved:

$$\partial_\beta\left(T^{\alpha\beta}_{\text{em}} + T^{\alpha\beta}_{\text{fluid}}\right) = 0.$$

In problems 18 and 19, we study the behavior of the electromagnetic field under rotations and Lorentz boosts. Under a general Lorentz transformation, $\Lambda^{\alpha'}{}_{\gamma}$, the electromagnetic field transforms according to:

$$F_{\alpha'\beta'} = F_{\gamma\delta}(\Lambda^{-1})^{\gamma}{}_{\alpha'}(\Lambda^{-1})^{\delta}{}_{\beta'}$$

18. First assume that the Lorentz transformation is a rotation about the z-axis:

$$\Lambda^{\alpha'}{}_{\gamma} = \begin{pmatrix} 1 & \vec{0} \\ \vec{0}^\top & R \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (\Lambda^{-1})^{\gamma}{}_{\alpha'} = \begin{pmatrix} 1 & \vec{0} \\ \vec{0}^\top & R^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where R is a 3×3 rotation matrix. Show that $\vec{E}$ and $\vec{B}$ transform as vectors:

$$E^{i'} = R^{i'}{}_j E^j \qquad\qquad B^{i'} = R^{i'}{}_j B^j.$$

Show that this generalizes to arbitrary rotation matrices.

19. Now assume that the Lorentz transformation is a boost in the z-direction with velocity $\vec{v} = v\hat{e}_z$:

$$\Lambda^{\alpha'}{}_{\gamma} = \begin{pmatrix} \cosh\lambda & 0 & 0 & \sinh\lambda \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\lambda & 0 & 0 & \cosh\lambda \end{pmatrix} \qquad (\Lambda^{-1})^{\gamma}{}_{\alpha'} = \begin{pmatrix} \cosh\lambda & 0 & 0 & -\sinh\lambda \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\lambda & 0 & 0 & \cosh\lambda \end{pmatrix}$$

where

$$\cosh\lambda = \gamma = \frac{1}{\sqrt{1-v^2}} \qquad\qquad \sinh\lambda = \gamma v = \frac{v}{\sqrt{1-v^2}}.$$

Find expressions for $\vec{E}'$ and $\vec{B}'$ in terms of $\vec{E}$ and $\vec{B}$ and either λ or v .

NOTE: **DO NOT TRY THE REMAINING PROBLEMS UNTIL YOU HAVE COMPLETED ALL OF THE PREVIOUS PROBLEMS**.

In problems 20 and 21, we study the Lagrangian and Hamiltonian formulations of electromagnetism. Each problem begins with a discussion of the analogous formulation of classical mechanics and the situation for a general field theory with fields ψ^A, for $A = 1,\ldots,N$. Then the special case of electromagnetism is treated with ψ^A replaced by A_α.

20. In classical mechanics, the Lagrangian is

$$L = T - V = \frac{1}{2}m|\vec{v}|^2 - V(\vec{x}).$$

In discussing this Lagrangian, it is useful to regard $\vec{x}$ and $\vec{v} = \frac{d\vec{x}}{dt}$ as independent variables. One then computes

$$\frac{\partial L}{\partial x^i} = \partial_i V \qquad\qquad p_i = \frac{\partial L}{\partial v^i} = mv_i.$$

(The quantity p_i is called the conjugate momentum to x^i.) Then the Euler-Lagrange equations for this Lagrangian are

$$\frac{d}{dt}\frac{\partial L}{\partial v^i} - \frac{\partial L}{\partial x^i} = 0$$

or

$$\frac{d}{dt}(mv_i) - \partial_i V = 0$$

which is Newton's equation with the force identified as $F_i = \partial_i V$, the gradient of the potential.

Similarly, in field theory, in discussing a Lagrangian density, L, it is useful to regard the fields ψ^A and their derivatives $\partial_\alpha \psi^A$ as independent variables. One then computes

$$\frac{\partial \mathrm{L}}{\partial \psi^A} \qquad\qquad \pi_A{}^\alpha = \frac{\partial \mathrm{L}}{\partial \partial_\alpha \psi^A}.$$

(The quantity $\pi_A \equiv \pi_A{}^0$ is called the conjugate momentum to ψ^A, while the 4-vector $\pi_A{}^\alpha$ is called the conjugate multi-momentum to ψ^A.) Then the Euler-Lagrange equations for the Lagrangian density, L, are

$$\partial_\alpha\left(\frac{\partial \mathrm{L}}{\partial \partial_\alpha \psi^A}\right) - \frac{\partial \mathrm{L}}{\partial \psi^A} = 0.$$

In class, I will derive both sets of Euler-Lagrange equations given above from appropriate variational principles. In this exercise, we apply the field theory version to the vacuum Maxwell Lagrangian density,

$$\mathrm{L} = -\frac{1}{4}F^{\gamma\delta}F_{\gamma\delta}.$$

Compute

$$\frac{\partial \mathrm{L}}{\partial A_\alpha} \qquad\qquad \pi^{\alpha\beta} = \frac{\partial \mathrm{L}}{\partial \partial_\beta A_\alpha}.$$

Identify the conjugate momenta to $A_0 = \phi$ and to A_i:

$$\pi^\alpha \equiv \pi^{\alpha 0} = \frac{\partial \mathrm{L}}{\partial \partial_0 A_\alpha}.$$

Compute the Euler-Lagrange equations

$$\partial_\beta\left(\frac{\partial \mathrm{L}}{\partial \partial_\beta A_\alpha}\right) - \frac{\partial \mathrm{L}}{\partial A_\alpha} = 0$$

and verify that these are the vacuum Maxwell equations.

HINTS: Explicitly write out all metrics in L but do not use the indices α or β and do not expand F in terms of derivatives of A. Use the product rule. Then compute the derivatives of F using formulas such as

$$\frac{\partial \partial_\delta A_\gamma}{\partial \partial_\beta A_\alpha} = \delta^\delta_\beta \delta^\gamma_\alpha.$$