MATH 304–503/504 Fall 2008

Sample problems for the final exam

Any problem may be altered or replaced by a different one!

Problem 1 (15 pts.) Find a quadratic polynomial $p(x)$ such that $p(-1) = p(3) = 6$ and $p'(2) = p(1)$.

Problem 2 (20 pts.) Consider a linear operator $L : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$L(\mathbf{x}) = (\mathbf{x} \cdot \mathbf{v}_1)\mathbf{v}_2, \quad \text{where } \mathbf{v}_1 = (1, 2, -1),\ \mathbf{v}_2 = (2, 2, 1).$$

(i) Find the matrix of the operator L.
(ii) Find the dimensions of the range and the kernel of L.
(iii) Find bases for the range and the kernel of L.

Problem 3 (20 pts.) Let $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 2 & 3 & 0 & 0 \end{pmatrix}$.

(i) Evaluate the determinant of the matrix A.
(ii) Find the inverse matrix A^{-1}.

Problem 4 (25 pts.) Let $B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

(i) Find all eigenvalues of the matrix B.
(ii) Find a basis for $\mathbb{R}^3$ consisting of eigenvectors of B.
(iii) Find an orthonormal basis for $\mathbb{R}^3$ consisting of eigenvectors of B.
(iv) Find a diagonal matrix X and an invertible matrix U such that $B = UXU^{-1}$.

Problem 5 (20 pts.) Let V be a subspace of $\mathbb{R}^4$ spanned by vectors $\mathbf{x}_1 = (1, 1, 0, 0)$, $\mathbf{x}_2 = (2, 0, -1, 1)$, and $\mathbf{x}_3 = (0, 1, 1, 0)$.

(i) Find the distance from the point $\mathbf{y} = (0, 0, 0, 4)$ to the subspace V.
(ii) Find the distance from the point $\mathbf{y}$ to the orthogonal complement $V^\perp$.

Bonus Problem 6 (20 pts.) (i) Find a matrix exponential $\exp(tC)$, where $C = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$ and $t \in \mathbb{R}$.

(ii) Solve a system of differential equations

$$\begin{cases} \dfrac{dx}{dt} = 3x + y, \\ \dfrac{dy}{dt} = 3y \end{cases}$$

subject to the initial conditions $x(0) = y(0) = 1$.

Bonus Problem 7 (20 pts.) Consider a linear operator $K : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$K(\mathbf{x}) = D\mathbf{x}, \quad \text{where} \quad D = \frac{1}{9} \begin{pmatrix} -4 & 7 & 4 \\ 1 & -4 & 8 \\ 8 & 4 & 1 \end{pmatrix}.$$

The operator K is a rotation about an axis.

(i) Find the axis of rotation.

(ii) Find the angle of rotation.