MATH 304

Linear Algebra

Lecture 4:
Row echelon form.
Gauss-Jordan reduction.

System of linear equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

Coefficient matrix ($m \times n$) and column vector of the right-hand sides ($m \times 1$):

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \qquad \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

System of linear equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

Augmented $m \times (n+1)$ *matrix:*

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right)$$

Solution of a system of linear equations splits into two parts: **(A)** elimination and **(B)** back substitution.

Both parts can be done by applying a finite number of **elementary operations**.

Since the elementary operations preserve the standard form of linear equations, we can trace the solution process by looking on the **augmented matrix**.

In terms of the augmented matrix, the elementary operations are **elementary row operations**.

Elementary operations for systems of linear equations:

(1) to multiply an equation by a nonzero scalar;

(2) to add an equation multiplied by a scalar to another equation;

(3) to interchange two equations.

Elementary row operations:

(1) to multiply a row by some $r \neq 0$;

(2) to add a scalar multiple of a row to another row;

(3) to interchange two rows.

Remark. The rows are added and multiplied by scalars as vectors (namely, **row vectors**).

Vector algebra

Let $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ be n-dimensional vectors, and $r \in \mathbb{R}$ be a scalar.

Vector sum: $\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$

Scalar multiple: $r\mathbf{a} = (ra_1, ra_2, \ldots, ra_n)$

Zero vector: $\mathbf{0} = (0, 0, \ldots, 0)$

Negative of a vector: $-\mathbf{b} = (-b_1, -b_2, \ldots, -b_n)$

Vector difference:

$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}) = (a_1 - b_1, a_2 - b_2, \ldots, a_n - b_n)$

Elementary row operations

Augmented matrix:

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ \hline a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \hline \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right) = \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_m \end{pmatrix},$$

where $\mathbf{v}_i = (a_{i1} \ a_{i2} \ \dots \ a_{in} \,|\, b_i)$ is a row vector.

Elementary row operations

Operation 1: to multiply the ith row by $r \neq 0$:

$$\begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_i \\ \vdots \\ \mathbf{v}_m \end{pmatrix} \longrightarrow \begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ r\mathbf{v}_i \\ \vdots \\ \mathbf{v}_m \end{pmatrix}$$

Elementary row operations

Operation 2: to add the ith row multiplied by r to the jth row:

$$\begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_i \\ \vdots \\ \mathbf{v}_j \\ \vdots \\ \mathbf{v}_m \end{pmatrix} \rightarrow \begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_i \\ \vdots \\ \mathbf{v}_j + r\mathbf{v}_i \\ \vdots \\ \mathbf{v}_m \end{pmatrix}$$

Elementary row operations

Operation 3: to interchange the ith row with the jth row:

$$\begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_i \\ \vdots \\ \mathbf{v}_j \\ \vdots \\ \mathbf{v}_m \end{pmatrix} \longrightarrow \begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_j \\ \vdots \\ \mathbf{v}_i \\ \vdots \\ \mathbf{v}_m \end{pmatrix}$$

Row echelon form

Definition. **Leading entry** of a matrix is the first nonzero entry in a row.

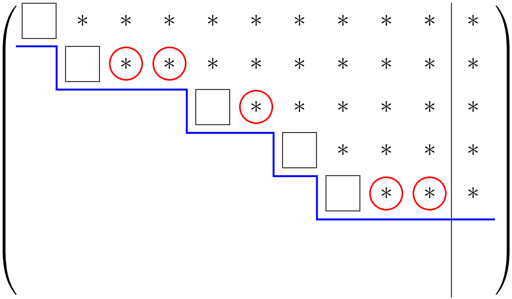
The goal of the Gaussian elimination is to convert the augmented matrix into **row echelon form**:

- leading entries shift to the right as we go from the first row to the last one;
- each leading entry is equal to 1.

$$\left(\begin{array}{cccccccccc|c} 1 & -1 & 3 & 0 & 2 & 1 & 4 & 0 & -3 & 7 & 2 \\ 0 & 1 & 1 & -2 & 0 & 0 & 6 & 1 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 3 & 1 & -4 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 9 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Row echelon form

General matrix in row echelon form:

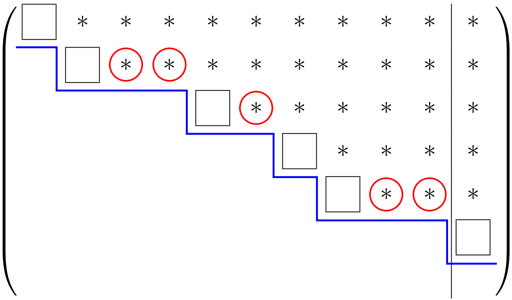


- leading entries are boxed (all equal to 1);
- all the entries below the staircase line are zero;
- each step of the staircase has height 1;
- each circle marks a free variable.

Strict triangular form is a particular case of row echelon form that can occur for systems of n equations in n variables:

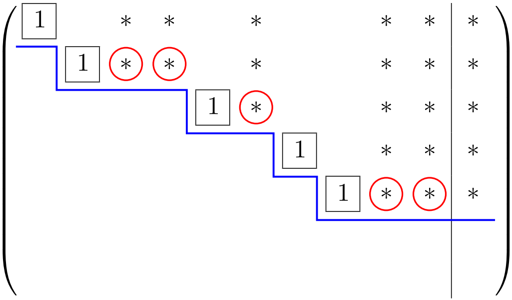
$$\left(\begin{array}{ccccccc|c}
\square & * & * & * & * & * & * & * \\
 & \square & * & * & * & * & * & * \\
 & & \square & * & * & * & * & * \\
 & & & \square & * & * & * & * \\
 & & & & \square & * & * & * \\
 & & & & & \square & * & * \\
 & & & & & & \square & *
\end{array}\right)$$

The original system of linear equations is **consistent** if there is no leading entry in the rightmost column of the augmented matrix in row echelon form.



Inconsistent system

The goal of the **Gauss-Jordan reduction** is to convert the augmented matrix into **reduced row echelon form**:



- all entries below the staircase line are zero;
- each boxed entry is 1, the other entries in its column are zero;
- each circle marks a free variable.

Example.

$$\begin{cases} x - y = 2 \\ 2x - y - z = 3 \\ x + y + z = 6 \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 2 & -1 & -1 & 3 \\ 1 & 1 & 1 & 6 \end{array}\right)$$

Row echelon form (also strict triangular):

$$\begin{cases} x - y = 2 \\ y - z = -1 \\ z = 2 \end{cases} \qquad \left(\begin{array}{ccc|c} \boxed{1} & -1 & 0 & 2 \\ 0 & \boxed{1} & -1 & -1 \\ 0 & 0 & \boxed{1} & 2 \end{array}\right)$$

Reduced row echelon form:

$$\begin{cases} x = 3 \\ y = 1 \\ z = 2 \end{cases} \qquad \left(\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 3 \\ 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & \boxed{1} & 2 \end{array}\right)$$

Another example.

$$\left\{\begin{array}{rcrcrcl} x & + & y & - & 2z & = & 1 \\ & & y & - & z & = & 3 \\ -x & + & 4y & - & 3z & = & 1 \end{array}\right. \qquad \left(\begin{array}{rrr|r} 1 & 1 & -2 & 1 \\ 0 & 1 & -1 & 3 \\ -1 & 4 & -3 & 1 \end{array}\right)$$

Row echelon form:

$$\left\{\begin{array}{rcrcrcl} x & + & y & - & 2z & = & 1 \\ & & y & - & z & = & 3 \\ & & & & 0 & = & 1 \end{array}\right. \qquad \left(\begin{array}{rrr|r} \boxed{1} & 1 & -2 & 1 \\ 0 & \boxed{1} & -1 & 3 \\ 0 & 0 & 0 & \boxed{1} \end{array}\right)$$

Reduced row echelon form:

$$\left\{\begin{array}{rcrcrcl} x & & & - & z & = & 0 \\ & & y & - & z & = & 0 \\ & & & & 0 & = & 1 \end{array}\right. \qquad \left(\begin{array}{rrr|r} \boxed{1} & 0 & -1 & 0 \\ 0 & \boxed{1} & -1 & 0 \\ 0 & 0 & 0 & \boxed{1} \end{array}\right)$$

Yet another example.

$$\left\{\begin{array}{rcrcrcr} x & + & y & - & 2z & = & 1 \\ & & y & - & z & = & 3 \\ -x & + & 4y & - & 3z & = & 14 \end{array}\right. \qquad \left(\begin{array}{rrr|r} 1 & 1 & -2 & 1 \\ 0 & 1 & -1 & 3 \\ -1 & 4 & -3 & 14 \end{array}\right)$$

Row echelon form:

$$\left\{\begin{array}{rcrcrcr} x & + & y & - & 2z & = & 1 \\ & & y & - & z & = & 3 \\ & & & & 0 & = & 0 \end{array}\right. \qquad \left(\begin{array}{rrr|r} \boxed{1} & 1 & -2 & 1 \\ 0 & \boxed{1} & -1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Reduced row echelon form:

$$\left\{\begin{array}{rcrcrcr} x & & & - & z & = & -2 \\ & & y & - & z & = & 3 \\ & & & & 0 & = & 0 \end{array}\right. \qquad \left(\begin{array}{rrr|r} \boxed{1} & 0 & -1 & -2 \\ 0 & \boxed{1} & -1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

New example. $\begin{cases} x_1 + 2x_2 + 3x_3 + 4x_4 = 10 \\ x_2 + 2x_3 + 3x_4 = 6 \end{cases}$

Augmented matrix: $\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 10 \\ 0 & 1 & 2 & 3 & 6 \end{array}\right)$

The matrix is already in row echelon form.
To convert it into reduced row echelon form,
add -2 times the 2nd row to the 1st row:

$$\left(\begin{array}{cccc|c} \boxed{1} & 0 & -1 & -2 & -2 \\ 0 & \boxed{1} & 2 & 3 & 6 \end{array}\right)$$

x_3 and x_4 are free variables

$$\begin{cases} x_1 - x_3 - 2x_4 = -2 \\ x_2 + 2x_3 + 3x_4 = 6 \end{cases} \iff \begin{cases} x_1 = x_3 + 2x_4 - 2 \\ x_2 = -2x_3 - 3x_4 + 6 \end{cases}$$

System of linear equations:

$$\begin{cases} x_1 + 2x_2 + 3x_3 + 4x_4 = 10 \\ x_2 + 2x_3 + 3x_4 = 6 \end{cases}$$

General solution:

$$\begin{cases} x_1 = t + 2s - 2 \\ x_2 = -2t - 3s + 6 \\ x_3 = t \\ x_4 = s \end{cases} \qquad (t, s \in \mathbb{R})$$

Example with a parameter.

$$\begin{cases} y + 3z = 0 \\ x + y - 2z = 0 \qquad (a \in \mathbb{R}) \\ x + 2y + az = 0 \end{cases}$$

The system is **homogeneous** (all right-hand sides are zeros). Therefore it is consistent ($x = y = z = 0$ is a solution).

Augmented matrix: $\left(\begin{array}{ccr|c} 0 & 1 & 3 & 0 \\ 1 & 1 & -2 & 0 \\ 1 & 2 & a & 0 \end{array}\right)$

Since the 1st row cannot serve as a pivotal one, we interchange it with the 2nd row:

$$\left(\begin{array}{ccc|c} 0 & 1 & 3 & 0 \\ 1 & 1 & -2 & 0 \\ 1 & 2 & a & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 1 & 2 & a & 0 \end{array}\right)$$

Now we can start the elimination.

First subtract the 1st row from the 3rd row:

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 1 & 2 & a & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & a+2 & 0 \end{array}\right)$$

The 2nd row is our new pivotal row.

Subtract the 2nd row from the 3rd row:

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & a+2 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & a-1 & 0 \end{array}\right)$$

At this point row reduction splits into two cases.

Case 1: $a \neq 1$. In this case, multiply the 3rd row by $(a-1)^{-1}$:

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & a-1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} \boxed{1} & 1 & -2 & 0 \\ 0 & \boxed{1} & 3 & 0 \\ 0 & 0 & \boxed{1} & 0 \end{array}\right)$$

The matrix is converted into row echelon form.
We proceed towards reduced row echelon form.

Subtract 3 times the 3rd row from the 2nd row:

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

Add 2 times the 3rd row to the 1st row:

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

Finally, subtract the 2nd row from the 1st row:

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \end{array}\right)$$

Thus $x = y = z = 0$ is the only solution.

Case 2: $a = 1$. In this case, the matrix is already in row echelon form:

$$\left(\begin{array}{ccc|c} \boxed{1} & 1 & -2 & 0 \\ 0 & \boxed{1} & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

To get reduced row echelon form, subtract the 2nd row from the 1st row:

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} \boxed{1} & 0 & -5 & 0 \\ 0 & \boxed{1} & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

z is a free variable.

$$\begin{cases} x - 5z = 0 \\ y + 3z = 0 \end{cases} \iff \begin{cases} x = 5z \\ y = -3z \end{cases}$$

System of linear equations:

$$\begin{cases} y + 3z = 0 \\ x + y - 2z = 0 \\ x + 2y + az = 0 \end{cases}$$

Solution: If $a \neq 1$ then $(x, y, z) = (0, 0, 0)$;
if $a = 1$ then $(x, y, z) = (5t, -3t, t)$, $t \in \mathbb{R}$.