MATH 304

Linear Algebra

Lecture 7:
Inverse matrix (continued).

Diagonal matrices

Definition. A square matrix is called **diagonal** if all non-diagonal entries are zeros.

Example. $\begin{pmatrix} 7 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, denoted $\operatorname{diag}(7, 1, 2)$.

Theorem Let $A = \operatorname{diag}(s_1, s_2, \ldots, s_n)$, $B = \operatorname{diag}(t_1, t_2, \ldots, t_n)$.

Then $A + B = \operatorname{diag}(s_1 + t_1, s_2 + t_2, \ldots, s_n + t_n)$,

$$rA = \operatorname{diag}(rs_1, rs_2, \ldots, rs_n).$$

$$AB = \operatorname{diag}(s_1 t_1, s_2 t_2, \ldots, s_n t_n).$$

Identity matrix

Definition. The **identity matrix** (or **unit matrix**) is a diagonal matrix with all diagonal entries equal to 1.

$$I_1 = (1), \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In general, $I = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}.$

Theorem. Let A be an arbitrary $m \times n$ matrix. Then $I_m A = A I_n = A$.

Inverse matrix

Definition. Let A be an $n\times n$ matrix. The **inverse** of A is an $n\times n$ matrix, denoted A^{-1}, such that

$$\boxed{AA^{-1} = A^{-1}A = I.}$$

If A^{-1} exists then the matrix A is called **invertible**. Otherwise A is called **singular**.

Let A and B be $n\times n$ matrices. If A is invertible then we can **divide** B by A:

$$\textit{left division: } A^{-1}B, \quad \textit{right division: } BA^{-1}.$$

Basic properties of inverse matrices:

- The inverse matrix (if it exists) is unique.
- If A is invertible, so is A^{-1}, and $(A^{-1})^{-1} = A$.
- If $n \times n$ matrices A and B are invertible, so is AB, and $(AB)^{-1} = B^{-1}A^{-1}$.
- If $n \times n$ matrices $A_1, A_2, \ldots, A_k$ are invertible, so is $A_1A_2 \ldots A_k$, and $(A_1A_2 \ldots A_k)^{-1} = A_k^{-1} \ldots A_2^{-1}A_1^{-1}$.

Inverting diagonal matrices

Theorem A diagonal matrix $D = \mathrm{diag}(d_1, \ldots, d_n)$ is invertible if and only if all diagonal entries are nonzero: $d_i \neq 0$ for $1 \leq i \leq n$.

If D is invertible then $D^{-1} = \mathrm{diag}(d_1^{-1}, \ldots, d_n^{-1})$.

$$\begin{pmatrix} d_1 & 0 & \ldots & 0 \\ 0 & d_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & d_n \end{pmatrix}^{-1} = \begin{pmatrix} d_1^{-1} & 0 & \ldots & 0 \\ 0 & d_2^{-1} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & d_n^{-1} \end{pmatrix}$$

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If D is invertible then $D^{-1} = \mathrm{diag}(d_1^{-1}, \ldots, d_n^{-1})$.

Proof: If all $d_i \neq 0$ then, clearly,

$$\mathrm{diag}(d_1, \ldots, d_n)\,\mathrm{diag}(d_1^{-1}, \ldots, d_n^{-1}) = \mathrm{diag}(1, \ldots, 1) = I,$$

$$\mathrm{diag}(d_1^{-1}, \ldots, d_n^{-1})\,\mathrm{diag}(d_1, \ldots, d_n) = \mathrm{diag}(1, \ldots, 1) = I.$$

Now suppose that $d_i = 0$ for some i. Then for any $n \times n$ matrix B the ith row of the matrix DB is a zero row. Hence $DB \neq I$.

Inverting 2-by-2 matrices

Definition. The **determinant** of a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\det A = ad - bc$.

Theorem A matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible if and only if $\det A \neq 0$.

If $\det A \neq 0$ then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Theorem A matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible if and only if $\det A \neq 0$. If $\det A \neq 0$ then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Proof: Let $B = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Then

$$AB = BA = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} = (ad - bc)I_2.$$

In the case $\det A \neq 0$, we have $A^{-1} = (\det A)^{-1}B$.
In the case $\det A = 0$, the matrix A is not invertible as otherwise $AB = O \implies A^{-1}AB = A^{-1}O \implies B = O$ $\implies A = O$, but the zero matrix is singular.

Problem. Solve a system $\begin{cases} 4x + 3y = 5, \\ 3x + 2y = -1. \end{cases}$

This system is equivalent to a matrix equation

$$\begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}.$$

Let $A = \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix}$. We have $\det A = -1 \neq 0$.

Hence A is invertible. Let's multiply both sides of the matrix equation by A^{-1} from the left:

$$\begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 5 \\ -1 \end{pmatrix},$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 5 \\ -1 \end{pmatrix} = \frac{1}{-1} \begin{pmatrix} 2 & -3 \\ -3 & 4 \end{pmatrix} \begin{pmatrix} 5 \\ -1 \end{pmatrix} = \begin{pmatrix} -13 \\ 19 \end{pmatrix}.$$

System of n linear equations in n variables:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases} \iff A\mathbf{x} = \mathbf{b},$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

Theorem If the matrix A is invertible then the system has a unique solution, which is $\mathbf{x} = A^{-1}\mathbf{b}$.

Problem. Solve the matrix equation $XA + B = X$, where $A = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 5 & 2 \\ 3 & 0 \end{pmatrix}$.

Since B is a 2×2 matrix, it follows that XA and X are also 2×2 matrices.

$$XA + B = X \iff X - XA = B$$
$$\iff X(I - A) = B \iff X = B(I - A)^{-1}$$

provided that $I - A$ is an invertible matrix.

$I - A = \begin{pmatrix} -3 & 2 \\ -1 & 0 \end{pmatrix}$,

- $I - A = \begin{pmatrix} -3 & 2 \\ -1 & 0 \end{pmatrix}$,
- $\det(I - A) = (-3) \cdot 0 - 2 \cdot (-1) = 2$,
- $(I - A)^{-1} = \frac{1}{2} \begin{pmatrix} 0 & -2 \\ 1 & -3 \end{pmatrix}$,
- $X = B(I - A)^{-1} = \begin{pmatrix} 5 & 2 \\ 3 & 0 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 0 & -2 \\ 1 & -3 \end{pmatrix}$

$$= \frac{1}{2} \begin{pmatrix} 5 & 2 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 0 & -2 \\ 1 & -3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 & -16 \\ 0 & -6 \end{pmatrix} = \begin{pmatrix} 1 & -8 \\ 0 & -3 \end{pmatrix}.$$

Fundamental results on inverse matrices

Theorem 1 Given a square matrix A, the following are equivalent:

(i) A is invertible;
(ii) $\mathbf{x} = \mathbf{0}$ is the only solution of the matrix equation $A\mathbf{x} = \mathbf{0}$;
(iii) the row echelon form of A has no zero rows;
(iv) the reduced row echelon form of A is the identity matrix.

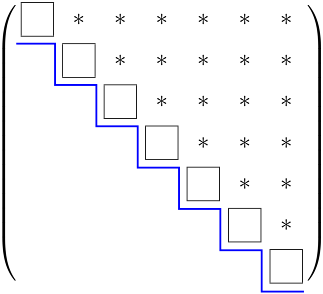
Theorem 2 Suppose that a sequence of elementary row operations converts a matrix A into the identity matrix.

Then the same sequence of operations converts the identity matrix into the inverse matrix A^{-1}.

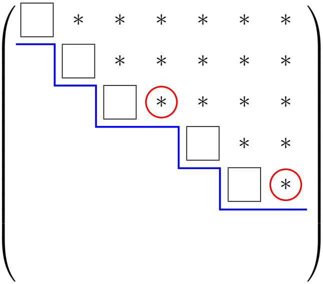
Theorem 3 For any $n \times n$ matrices A and B,

$$BA = I \iff AB = I.$$

Row echelon form of a square matrix:



invertible case



noninvertible case