MATH 304

Linear Algebra

Lecture 14: Linear independence (continued).

Linear independence

Definition. Let V be a vector space. Vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in V$ are called **linearly dependent** if they satisfy a relation

$$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k = \mathbf{0},$$

where the coefficients $r_1, \dots, r_k \in \mathbb{R}$ are not all equal to zero. Otherwise the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are called **linearly independent**. That is, if

$$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k = \mathbf{0} \implies r_1 = \cdots = r_k = 0.$$

An infinite set $S \subset V$ is **linearly dependent** if there are some linearly dependent vectors $\mathbf{v}_1, \dots, \mathbf{v}_k \in S$. Otherwise S is **linearly independent**.

Theorem Vectors $\mathbf{v}_1, \dots, \mathbf{v}_k \in V$ are linearly dependent if and only if one of them is a linear combination of the other $k-1$ vectors.

Examples of linear independence.

- Vectors $\mathbf{e}_1 = (1,0,0)$, $\mathbf{e}_2 = (0,1,0)$, and $\mathbf{e}_3 = (0,0,1)$ in $\mathbb{R}^3$.

- Matrices $E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and $E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

Examples of linear independence

- Polynomials $1, x, x^2, \ldots, x^n$.

$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0$ identically
$\implies a_i = 0$ for $0 \le i \le n$

- The infinite set $\{1, x, x^2, \ldots, x^n, \ldots\}$.

- Polynomials $p_1(x) = 1$, $p_2(x) = x - 1$, and $p_3(x) = (x-1)^2$.

$a_1 p_1(x) + a_2 p_2(x) + a_3 p_3(x) = a_1 + a_2(x-1) + a_3(x-1)^2 =$
$= (a_1 - a_2 + a_3) + (a_2 - 2a_3)x + a_3 x^2$.

Hence $a_1 p_1(x) + a_2 p_2(x) + a_3 p_3(x) = 0$ identically
$\implies a_1 - a_2 + a_3 = a_2 - 2a_3 = a_3 = 0$
$\implies a_1 = a_2 = a_3 = 0$

Problem Let $\mathbf{v}_1 = (1, 2, 0)$, $\mathbf{v}_2 = (3, 1, 1)$, and $\mathbf{v}_3 = (4, -7, 3)$. Determine whether vectors $\mathbf{v}_2, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent.

We have to check if there exist $r_1, r_2, r_3 \in \mathbb{R}$ not all zero such that $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + r_3\mathbf{v}_3 = \mathbf{0}$.

This vector equation is equivalent to a system

$$\begin{cases} r_1 + 3r_2 + 4r_3 = 0 \\ 2r_1 + r_2 - 7r_3 = 0 \\ 0r_1 + r_2 + 3r_3 = 0 \end{cases} \qquad \left(\begin{array}{ccr|c} 1 & 3 & 4 & 0 \\ 2 & 1 & -7 & 0 \\ 0 & 1 & 3 & 0 \end{array}\right)$$

The vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent if and only if the matrix $A = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ is singular.
We obtain that $\det A = 0$.

Theorem Vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m \in \mathbb{R}^n$ are linearly dependent whenever $m > n$ (i.e., the number of coordinates is less than the number of vectors).

Proof: Let $\mathbf{v}_j = (a_{1j}, a_{2j}, \dots, a_{nj})$ for $j = 1, 2, \dots, m$. Then the vector equality $t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_m\mathbf{v}_m = \mathbf{0}$ is equivalent to the system

$$\begin{cases} a_{11}t_1 + a_{12}t_2 + \cdots + a_{1m}t_m = 0, \\ a_{21}t_1 + a_{22}t_2 + \cdots + a_{2m}t_m = 0, \\ \qquad\qquad \cdots\cdots\cdots \\ a_{n1}t_1 + a_{n2}t_2 + \cdots + a_{nm}t_m = 0. \end{cases}$$

Note that vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ are columns of the matrix (a_{ij}). The number of leading entries in the row echelon form is at most n. If $m > n$ then there are free variables, therefore the zero solution is not unique.

Example. Consider vectors $\mathbf{v}_1 = (1, -1, 1)$, $\mathbf{v}_2 = (1, 0, 0)$, $\mathbf{v}_3 = (1, 1, 1)$, and $\mathbf{v}_4 = (1, 2, 4)$ in $\mathbb{R}^3$.

Two vectors are linearly dependent if and only if they are parallel. Hence $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.

Vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent if and only if the matrix $A = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ is invertible.

$$\det A = \begin{vmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{vmatrix} = - \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = 2 \neq 0.$$

Therefore $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent.

Four vectors in $\mathbb{R}^3$ are always linearly dependent.

Thus $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are linearly dependent.

Problem 1. Show that functions 1, e^x, and e^{-x} are linearly independent in $F(\mathbb{R})$.

Proof: Suppose that $a + be^x + ce^{-x} = 0$ for some $a, b, c \in \mathbb{R}$. We have to show that $a = b = c = 0$.

$$x = 0 \implies a + b + c = 0$$
$$x = 1 \implies a + be + ce^{-1} = 0$$
$$x = -1 \implies a + be^{-1} + ce = 0$$

The matrix of the system is $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & e & e^{-1} \\ 1 & e^{-1} & e \end{pmatrix}$.

$$\det A = e^2 - e^{-2} - 2e + 2e^{-1} =$$
$$= (e - e^{-1})(e + e^{-1}) - 2(e - e^{-1}) =$$
$$= (e-e^{-1})(e+e^{-1}-2) = (e-e^{-1})(e^{1/2}-e^{-1/2})^2 \neq 0.$$

Hence the system has a unique solution $a = b = c = 0$.

Problem 2. Show that functions e^x, e^{2x}, and e^{3x} are linearly independent in $C^\infty(\mathbb{R})$.

Suppose that $ae^x + be^{2x} + ce^{3x} = 0$ for all $x \in \mathbb{R}$, where a, b, c are constants. We have to show that $a = b = c = 0$.

Differentiate this identity twice:

$$ae^x + 2be^{2x} + 3ce^{3x} = 0,$$
$$ae^x + 4be^{2x} + 9ce^{3x} = 0.$$

It follows that $A\mathbf{v} = \mathbf{0}$, where

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} ae^x \\ be^{2x} \\ ce^{3x} \end{pmatrix}.$$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} ae^x \\ be^{2x} \\ ce^{3x} \end{pmatrix}.$$

To compute $\det A$, subtract the 1st row from the 2nd and the 3rd rows:

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 4 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 8 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 3 & 8 \end{vmatrix} = 2.$$

Since A is invertible, we obtain

$$A\mathbf{v} = \mathbf{0} \implies \mathbf{v} = \mathbf{0} \implies ae^x = be^{2x} = ce^{3x} = 0$$
$$\implies a = b = c = 0$$

Theorem 1 Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be distinct real numbers. Then the functions $e^{\lambda_1 x}, e^{\lambda_2 x}, \ldots, e^{\lambda_k x}$ are linearly independent.

Theorem 2 The set of functions

$$\{x^m e^{\lambda x} \mid \lambda \in \mathbb{R},\ m = 0, 1, 2, \ldots\}$$

is linearly independent.