MATH 304
Linear Algebra

Lecture 20:
Review for Test 1.

Topics for Test 1

Part I: Elementary linear algebra (Leon 1.1–1.4, 2.1–2.2)

- Systems of linear equations: elementary operations, Gaussian elimination, back substitution.

- Matrix of coefficients and augmented matrix. Elementary row operations, row echelon form and reduced row echelon form.

- Matrix algebra. Inverse matrix.

- Determinants: explicit formulas for 2×2 and 3×3 matrices, row and column expansions, elementary row and column operations.

Topics for Test 1

Part II: Abstract linear algebra (Leon 3.1–3.6)

- Vector spaces (vectors, matrices, polynomials, functional spaces).
- Subspaces. Nullspace, column space, and row space of a matrix.
- Span, spanning set. Linear independence.
- Bases and dimension. Rank and nullity of a matrix.
- Change of coordinates, transition matrix.

Sample problems for Test 1

Problem 1 (20 pts.) Find the point of intersection of the planes $x + 2y - z = 1$, $x - 3y = -5$, and $2x + y + z = 0$ in $\mathbb{R}^3$.

Problem 2 (30 pts.) Let $A = \begin{pmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 2 & 0 & -1 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix}$.

(i) Evaluate the determinant of the matrix A.
(ii) Find the inverse matrix A^{-1}.

Problem 3 (20 pts.) Determine which of the following subsets of $\mathbb{R}^3$ are subspaces. Briefly explain.

(i) The set S_1 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $xyz = 0$.
(ii) The set S_2 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $x + y + z = 0$.
(iii) The set S_3 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $y^2 + z^2 = 0$.
(iv) The set S_4 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $y^2 - z^2 = 0$.

Problem 4 (30 pts.) Let $B = \begin{pmatrix} 0 & -1 & 4 & 1 \\ 1 & 1 & 2 & -1 \\ -3 & 0 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix}$.

(i) Find the rank and the nullity of the matrix B.
(ii) Find a basis for the row space of B, then extend this basis to a basis for $\mathbb{R}^4$.

Bonus Problem 5 (20 pts.) Show that the functions $f_1(x) = x$, $f_2(x) = xe^x$, and $f_3(x) = e^{-x}$ are linearly independent in the vector space $C^\infty(\mathbb{R})$.

Bonus Problem 6 (20 pts.) Let V and W be subspaces of the vector space $\mathbb{R}^n$ such that $V \cup W$ is also a subspace of $\mathbb{R}^n$. Show that $V \subset W$ or $W \subset V$.

Problem 1. Find the point of intersection of the planes $x + 2y - z = 1$, $x - 3y = -5$, and $2x + y + z = 0$ in $\mathbb{R}^3$.

The intersection point (x, y, z) is a solution of the system

$$\begin{cases} x + 2y - z = 1, \\ x - 3y = -5, \\ 2x + y + z = 0. \end{cases}$$

To solve the system, we convert its augmented matrix into reduced row echelon form using elementary row operations:

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 1 & -3 & 0 & -5 \\ 2 & 1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -5 & 1 & -6 \\ 2 & 1 & 1 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{rrr|r} 1 & 2 & -1 & 1 \\ 0 & -5 & 1 & -6 \\ 2 & 1 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{rrr|r} 1 & 2 & -1 & 1 \\ 0 & -5 & 1 & -6 \\ 0 & -3 & 3 & -2 \end{array}\right) \longrightarrow \left(\begin{array}{rrr|r} 1 & 2 & -1 & 1 \\ 0 & -3 & 3 & -2 \\ 0 & -5 & 1 & -6 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{rrr|r} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & \frac{2}{3} \\ 0 & -5 & 1 & -6 \end{array}\right) \longrightarrow \left(\begin{array}{rrr|r} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & \frac{2}{3} \\ 0 & 0 & -4 & -\frac{8}{3} \end{array}\right) \longrightarrow \left(\begin{array}{rrr|r} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & \frac{2}{3} \\ 0 & 0 & 1 & \frac{2}{3} \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{rrr|r} 1 & 2 & -1 & 1 \\ 0 & 1 & 0 & \frac{4}{3} \\ 0 & 0 & 1 & \frac{2}{3} \end{array}\right) \longrightarrow \left(\begin{array}{rrr|r} 1 & 2 & 0 & \frac{5}{3} \\ 0 & 1 & 0 & \frac{4}{3} \\ 0 & 0 & 1 & \frac{2}{3} \end{array}\right) \longrightarrow \left(\begin{array}{rrr|r} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & \frac{4}{3} \\ 0 & 0 & 1 & \frac{2}{3} \end{array}\right).$$

Thus the three planes intersect at the point $(-1, \frac{4}{3}, \frac{2}{3})$.

Problem 1. Find the point of intersection of the planes $x + 2y - z = 1$, $x - 3y = -5$, and $2x + y + z = 0$ in $\mathbb{R}^3$.

Alternative solution: The intersection point (x, y, z) is a solution of the system

$$\begin{cases} x + 2y - z = 1, \\ x - 3y = -5, \\ 2x + y + z = 0. \end{cases}$$

Add all three equations: $4x = -4 \implies x = -1$.
Substitute $x = -1$ into the 2nd equation: $\implies y = \frac{4}{3}$.
Substitute $x = -1$ and $y = \frac{4}{3}$ into the 3rd equation:
$\implies z = \frac{2}{3}$.

It remains to check that $x = -1$, $y = \frac{4}{3}$, $z = \frac{2}{3}$ is indeed a solution of the system.

Problem 2. Let $A = \begin{pmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 2 & 0 & -1 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix}$.

(i) Evaluate the determinant of the matrix A.

Subtract 2 times the 4th column of A from the 1st column:

$$\begin{vmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 2 & 0 & -1 & 1 \\ 2 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} -1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix}.$$

Expand the determinant by the 4th row:

$$\begin{vmatrix} -1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} -1 & -2 & 4 \\ 2 & 3 & 2 \\ 0 & 0 & -1 \end{vmatrix}.$$

Expand the determinant by the 3rd row:

$$\begin{vmatrix} -1 & -2 & 4 \\ 2 & 3 & 2 \\ 0 & 0 & -1 \end{vmatrix} = (-1)\begin{vmatrix} -1 & -2 \\ 2 & 3 \end{vmatrix} = -1.$$

Problem 2. Let $A = \begin{pmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 2 & 0 & -1 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix}$.

(ii) Find the inverse matrix A^{-1}.

First we merge the matrix A with the identity matrix into one 4×8 matrix

$$(A \,|\, I) = \left(\begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 2 & 3 & 2 & 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right).$$

Then we apply elementary row operations to this matrix until the left part becomes the identity matrix.

Subtract 2 times the 1st row from the 2nd row:

$$\left(\begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 7 & -6 & -2 & -2 & 1 & 0 & 0 \\ 2 & 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

Subtract 2 times the 1st row from the 3rd row:

$$\left(\begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 7 & -6 & -2 & -2 & 1 & 0 & 0 \\ 0 & 4 & -9 & -1 & -2 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

Subtract 2 times the 1st row from the 4th row:

$$\left(\begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 7 & -6 & -2 & -2 & 1 & 0 & 0 \\ 0 & 4 & -9 & -1 & -2 & 0 & 1 & 0 \\ 0 & 4 & -8 & -1 & -2 & 0 & 0 & 1 \end{array}\right)$$

Subtract 2 times the 4th row from the 2nd row:

$$\left(\begin{array}{rrrr|rrrr} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 10 & 0 & 2 & 1 & 0 & -2 \\ 0 & 4 & -9 & -1 & -2 & 0 & 1 & 0 \\ 0 & 4 & -8 & -1 & -2 & 0 & 0 & 1 \end{array}\right)$$

Subtract the 4th row from the 3rd row:

$$\left(\begin{array}{rrrr|rrrr} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 10 & 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 4 & -8 & -1 & -2 & 0 & 0 & 1 \end{array}\right)$$

Add 4 times the 2nd row to the 4th row:

$$\left(\begin{array}{rrrr|rrrr} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 10 & 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 32 & -1 & 6 & 4 & 0 & -7 \end{array}\right)$$

Add 32 times the 3rd row to the 4th row:

$$\left(\begin{array}{rrrr|rrrr} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 10 & 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 6 & 4 & 32 & -39 \end{array}\right)$$

Add 10 times the 3rd row to the 2nd row:

$$\left(\begin{array}{rrrr|rrrr} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 2 & 1 & 10 & -12 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 6 & 4 & 32 & -39 \end{array}\right)$$

Add the 4th row to the 1st row:

$$\left(\begin{array}{rrrr|rrrr} 1 & -2 & 4 & 0 & 7 & 4 & 32 & -39 \\ 0 & -1 & 0 & 0 & 2 & 1 & 10 & -12 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 6 & 4 & 32 & -39 \end{array}\right)$$

Add 4 times the 3rd row to the 1st row:

$$\left(\begin{array}{rrrr|rrrr} 1 & -2 & 0 & 0 & 7 & 4 & 36 & -43 \\ 0 & -1 & 0 & 0 & 2 & 1 & 10 & -12 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 6 & 4 & 32 & -39 \end{array}\right)$$

Subtract 2 times the 2nd row from the 1st row:

$$\left(\begin{array}{rrrr|rrrr} 1 & 0 & 0 & 0 & 3 & 2 & 16 & -19 \\ 0 & -1 & 0 & 0 & 2 & 1 & 10 & -12 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 6 & 4 & 32 & -39 \end{array}\right)$$

Multiply the 2nd, the 3rd, and the 4th rows by -1:

$$\left(\begin{array}{rrrr|rrrr} 1 & 0 & 0 & 0 & 3 & 2 & 16 & -19 \\ 0 & 1 & 0 & 0 & -2 & -1 & -10 & 12 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & -6 & -4 & -32 & 39 \end{array}\right)$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 3 & 2 & 16 & -19 \\ 0 & 1 & 0 & 0 & -2 & -1 & -10 & 12 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & -6 & -4 & -32 & 39 \end{array}\right) = (I \mid A^{-1})$$

Finally the left part of our 4 × 8 matrix is transformed into the identity matrix. Therefore the current right part is the inverse matrix of A. Thus

$$A^{-1} = \begin{pmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 2 & 0 & -1 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & 2 & 16 & -19 \\ -2 & -1 & -10 & 12 \\ 0 & 0 & -1 & 1 \\ -6 & -4 & -32 & 39 \end{pmatrix}.$$

Problem 2. Let $A = \begin{pmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 2 & 0 & -1 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix}$.

(i) Evaluate the determinant of the matrix A.

Alternative solution: We have transformed A into the identity matrix using elementary row operations. These included no row exchanges and three row multiplications, each time by -1.

It follows that $\det I = (-1)^3 \det A$.
$\implies \quad \det A = -\det I = -1$.

Problem 3. Determine which of the following subsets of $\mathbb{R}^3$ are subspaces. Briefly explain.

A subset of $\mathbb{R}^3$ is a subspace if it is closed under addition and scalar multiplication. Besides, the subset must not be empty.

(i) The set S_1 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $xyz = 0$.

$(0, 0, 0) \in S_1 \implies S_1$ is not empty.

$xyz = 0 \implies (rx)(ry)(rz) = r^3 xyz = 0.$
That is, $\mathbf{v} = (x, y, z) \in S_1 \implies r\mathbf{v} = (rx, ry, rz) \in S_1.$
Hence S_1 is closed under scalar multiplication.

However S_1 is not closed under addition.
Counterexample: $(1, 1, 0) + (0, 0, 1) = (1, 1, 1).$

Problem 3. Determine which of the following subsets of $\mathbb{R}^3$ are subspaces. Briefly explain.

A subset of $\mathbb{R}^3$ is a subspace if it is closed under addition and scalar multiplication. Besides, the subset must not be empty.

(ii) The set S_2 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $x + y + z = 0$.

$(0, 0, 0) \in S_2 \implies S_2$ is not empty.

$x + y + z = 0 \implies rx + ry + rz = r(x + y + z) = 0.$
Hence S_2 is closed under scalar multiplication.

$x + y + z = x' + y' + z' = 0 \implies$
$(x+x')+(y+y')+(z+z') = (x+y+z)+(x'+y'+z') = 0.$
That is, $\mathbf{v} = (x, y, z)$, $\mathbf{v}' = (x', y', z') \in S_2$
$\implies \mathbf{v} + \mathbf{v}' = (x + x', y + y', z + z') \in S_2.$
Hence S_2 is closed under addition.

(iii) The set S_3 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $y^2 + z^2 = 0$.

$y^2 + z^2 = 0 \iff y = z = 0.$

S_3 is a nonempty set closed under addition and scalar multiplication.

(iv) The set S_4 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $y^2 - z^2 = 0$.

S_4 is a nonempty set closed under scalar multiplication.
However S_4 is not closed under addition.
Counterexample: $(0, 1, 1) + (0, 1, -1) = (0, 2, 0)$.

Problem 4. Let $B = \begin{pmatrix} 0 & -1 & 4 & 1 \\ 1 & 1 & 2 & -1 \\ -3 & 0 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix}$.

(i) Find the rank and the nullity of the matrix B.

The rank (= dimension of the row space) and the nullity (= dimension of the nullspace) of a matrix are preserved under elementary row operations. We apply such operations to convert the matrix B into row echelon form.

Interchange the 1st row with the 2nd row:

$$\to \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & -1 & 4 & 1 \\ -3 & 0 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix}$$

Add 3 times the 1st row to the 3rd row, then subtract 2 times the 1st row from the 4th row:

$$\rightarrow \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & -1 & 4 & 1 \\ 0 & 3 & 5 & -3 \\ 2 & -1 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & -1 & 4 & 1 \\ 0 & 3 & 5 & -3 \\ 0 & -3 & -4 & 3 \end{pmatrix}$$

Multiply the 2nd row by -1:

$$\rightarrow \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 3 & 5 & -3 \\ 0 & -3 & -4 & 3 \end{pmatrix}$$

Add the 4th row to the 3rd row:

$$\rightarrow \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -3 & -4 & 3 \end{pmatrix}$$

Add 3 times the 2nd row to the 4th row:

$$\rightarrow \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -16 & 0 \end{pmatrix}$$

Add 16 times the 3rd row to the 4th row:

$$\rightarrow \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Now that the matrix is in row echelon form, its rank equals the number of nonzero rows, which is 3. Since

$$(\text{rank of } B) + (\text{nullity of } B) = (\text{the number of columns of } B) = 4,$$

it follows that the nullity of B equals 1.

Problem 4. Let $B = \begin{pmatrix} 0 & -1 & 4 & 1 \\ 1 & 1 & 2 & -1 \\ -3 & 0 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix}$.

(ii) Find a basis for the row space of B, then extend this basis to a basis for $\mathbb{R}^4$.

The row space of a matrix is invariant under elementary row operations. Therefore the row space of the matrix B is the same as the row space of its row echelon form:

$$\begin{pmatrix} 0 & -1 & 4 & 1 \\ 1 & 1 & 2 & -1 \\ -3 & 0 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The nonzero rows of the latter matrix are linearly independent so that they form a basis for its row space:

$$\mathbf{v}_1 = (1,1,2,-1), \;\; \mathbf{v}_2 = (0,1,-4,-1), \;\; \mathbf{v}_3 = (0,0,1,0).$$

To extend the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to a basis for $\mathbb{R}^4$, we need a vector $\mathbf{v}_4 \in \mathbb{R}^4$ that is not a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

It is known that at least one of the vectors $\mathbf{e}_1 = (1,0,0,0)$, $\mathbf{e}_2 = (0,1,0,0)$, $\mathbf{e}_3 = (0,0,1,0)$, and $\mathbf{e}_4 = (0,0,0,1)$ can be chosen as $\mathbf{v}_4$.

In particular, the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{e}_4$ form a basis for $\mathbb{R}^4$. This follows from the fact that the 4×4 matrix whose rows are these vectors is not singular:

$$\begin{vmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1 \neq 0.$$

Bonus Problem 5. Show that the functions $f_1(x) = x$, $f_2(x) = xe^x$, and $f_3(x) = e^{-x}$ are linearly independent in the vector space $C^\infty(\mathbb{R})$.

Suppose that $af_1(x) + bf_2(x) + cf_3(x) = 0$ for all $x \in \mathbb{R}$, where a, b, c are constants. We have to show that $a = b = c = 0$.

Let us differentiate the identity 4 times:

$$\begin{gathered} ax + bxe^x + ce^{-x} = 0, \\ a + be^x + bxe^x - ce^{-x} = 0, \\ 2be^x + bxe^x + ce^{-x} = 0, \\ 3be^x + bxe^x - ce^{-x} = 0, \\ 4be^x + bxe^x + ce^{-x} = 0. \end{gathered}$$

(the 5th identity)−(the 3rd identity): $2be^x = 0 \implies b = 0$.

Substitute $b = 0$ in the 3rd identity: $ce^{-x} = 0 \implies c = 0$.

Substitute $b = c = 0$ in the 2nd identity: $a = 0$.

Bonus Problem 5. Show that the functions $f_1(x) = x$, $f_2(x) = xe^x$, and $f_3(x) = e^{-x}$ are linearly independent in the vector space $C^\infty(\mathbb{R})$.

Alternative solution: Suppose that $ax + bxe^x + ce^{-x} = 0$ for all $x \in \mathbb{R}$, where a, b, c are constants. We have to show that $a = b = c = 0$.

For any $x \neq 0$ divide both sides of the identity by xe^x:

$$ae^{-x} + b + cx^{-1}e^{-2x} = 0.$$

The left-hand side approaches b as $x \to +\infty$. $\implies b = 0$

Now $ax + ce^{-x} = 0$ for all $x \in \mathbb{R}$. For any $x \neq 0$ divide both sides of the identity by x:

$$a + cx^{-1}e^{-x} = 0.$$

The left-hand side approaches a as $x \to +\infty$. $\implies a = 0$

Now $ce^{-x} = 0 \implies c = 0$.

Bonus Problem 6. Let V and W be subspaces of the vector space $\mathbb{R}^n$ such that $V \cup W$ is also a subspace of $\mathbb{R}^n$. Show that $V \subset W$ or $W \subset V$.

Assume the contrary: $V \not\subset W$ and $W \not\subset V$. Then there exist vectors $\mathbf{v} \in V$ and $\mathbf{w} \in W$ such that $\mathbf{v} \notin W$ and $\mathbf{w} \notin V$.

Let $\mathbf{x} = \mathbf{v} + \mathbf{w}$. Since $V \cup W$ is a subspace, we have

$$\mathbf{v}, \mathbf{w} \in V \cup W \implies \mathbf{x} \in V \cup W \implies \mathbf{x} \in V \text{ or } \mathbf{x} \in W.$$

Case 1: $\mathbf{x} \in V$. $\quad \implies \mathbf{x}, \mathbf{v} \in V \implies \mathbf{w} = \mathbf{x} - \mathbf{v} \in V.$

Case 2: $\mathbf{x} \in W$. $\quad \implies \mathbf{x}, \mathbf{w} \in W \implies \mathbf{v} = \mathbf{x} - \mathbf{w} \in W.$

In both cases, we get a **contradiction!**