MATH 304

Linear Algebra

Lecture 21:
General linear equations.
Matrix transformations.

Linear mapping = linear transformation = linear function

Definition. Given vector spaces V_1 and V_2, a mapping $L : V_1 \to V_2$ is **linear** if

$$\boxed{L(\mathbf{x}+\mathbf{y}) = L(\mathbf{x}) + L(\mathbf{y}),}$$

$$\boxed{L(r\mathbf{x}) = rL(\mathbf{x})}$$

for any $\mathbf{x}, \mathbf{y} \in V_1$ and $r \in \mathbb{R}$.

Basic properties of linear mappings:

- $L(r_1\mathbf{v}_1 + \cdots + r_k\mathbf{v}_k) = r_1L(\mathbf{v}_1) + \cdots + r_kL(\mathbf{v}_k)$ for all $k \geq 1$, $\mathbf{v}_1, \dots, \mathbf{v}_k \in V_1$, and $r_1, \dots, r_k \in \mathbb{R}$.
- $L(\mathbf{0}_1) = \mathbf{0}_2$, where $\mathbf{0}_1$ and $\mathbf{0}_2$ are zero vectors in V_1 and V_2, respectively.
- $L(-\mathbf{v}) = -L(\mathbf{v})$ for any $\mathbf{v} \in V_1$.

Range and kernel

Let V, W be vector spaces and $L : V \to W$ be a linear mapping.

Definition. The **range** (or **image**) of L is the set of all vectors $\mathbf{w} \in W$ such that $\mathbf{w} = L(\mathbf{v})$ for some $\mathbf{v} \in V$. The range of L is denoted $L(V)$.

The **kernel** of L, denoted $\ker L$, is the set of all vectors $\mathbf{v} \in V$ such that $L(\mathbf{v}) = \mathbf{0}$.

Theorem **(i)** The range of L is a subspace of W.
(ii) The kernel of L is a subspace of V.

General linear equations

Definition. A **linear equation** is an equation of the form

$$\boxed{L(\mathbf{x}) = \mathbf{b},}$$

where $L : V \to W$ is a linear mapping, $\mathbf{b}$ is a given vector from W, and $\mathbf{x}$ is an unknown vector from V.

The range of L is the set of all vectors $\mathbf{b} \in W$ such that the equation $L(\mathbf{x}) = \mathbf{b}$ has a solution.

The kernel of L is the solution set of the **homogeneous** linear equation $L(\mathbf{x}) = \mathbf{0}$.

Theorem If the linear equation $L(\mathbf{x}) = \mathbf{b}$ is solvable then the general solution is

$$\mathbf{x}_0 + t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k,$$

where $\mathbf{x}_0$ is a particular solution, $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is a basis for the kernel of L, and $t_1, \ldots, t_k$ are arbitrary scalars.

Example. $\begin{cases} x + y + z = 4, \\ x + 2y = 3. \end{cases}$

$$L : \mathbb{R}^3 \to \mathbb{R}^2, \quad L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Linear equation: $L(\mathbf{x}) = \mathbf{b}$, where $\mathbf{b} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & 2 & 0 & 3 \end{array}\right) \to \left(\begin{array}{ccc|c} \color{red}{1} & 1 & 1 & 4 \\ 0 & \color{red}{1} & -1 & -1 \end{array}\right) \to \left(\begin{array}{ccc|c} \color{red}{1} & 0 & 2 & 5 \\ 0 & \color{red}{1} & -1 & -1 \end{array}\right)$$

$$\begin{cases} x + 2z = 5 \\ y - z = -1 \end{cases} \iff \begin{cases} x = 5 - 2z \\ y = -1 + z \end{cases}$$

$$(x, y, z) = (5 - 2t, -1 + t, t) = (5, -1, 0) + t(-2, 1, 1).$$

Example. $u''(x) + u(x) = e^{2x}$.

Linear operator $L : C^2(\mathbb{R}) \to C(\mathbb{R})$, $Lu = u'' + u$.
Linear equation: $Lu = b$, where $b(x) = e^{2x}$.

It can be shown that the range of L is the entire space $C(\mathbb{R})$ while the kernel of L is spanned by the functions $\sin x$ and $\cos x$.

Observe that
$(Lb)(x) = b''(x)+b(x) = 4e^{2x}+e^{2x} = 5e^{2x} = 5b(x)$.
By linearity, $u_0 = \frac{1}{5}b$ is a particular solution.
Thus the general solution is

$$u(x) = \tfrac{1}{5}e^{2x} + t_1 \sin x + t_2 \cos x.$$

Matrix transformations

Any $m{\times}n$ matrix A gives rise to a transformation $L : \mathbb{R}^n \to \mathbb{R}^m$ given by $L(\mathbf{x}) = A\mathbf{x}$, where $\mathbf{x} \in \mathbb{R}^n$ and $L(\mathbf{x}) \in \mathbb{R}^m$ are regarded as column vectors. This transformation is **linear**.

Indeed, $L(\mathbf{x}+\mathbf{y}) = A(\mathbf{x}+\mathbf{y}) = A\mathbf{x} + A\mathbf{y} = L(\mathbf{x}) + L(\mathbf{y})$, $L(r\mathbf{x}) = A(r\mathbf{x}) = r(A\mathbf{x}) = rL(\mathbf{x})$.

Example. $$L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 4 & 7 \\ 0 & 5 & 8 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Let $\mathbf{e}_1 = (1,0,0)$, $\mathbf{e}_2 = (0,1,0)$, $\mathbf{e}_3 = (0,0,1)$ be the standard basis for $\mathbb{R}^3$. We have that $L(\mathbf{e}_1) = (1,3,0)$, $L(\mathbf{e}_2) = (0,4,5)$, $L(\mathbf{e}_3) = (2,7,8)$. Thus $L(\mathbf{e}_1), L(\mathbf{e}_2), L(\mathbf{e}_3)$ are columns of the matrix.

Problem. Find a linear mapping $L : \mathbb{R}^3 \to \mathbb{R}^2$ such that $L(\mathbf{e}_1) = (1, 1)$, $L(\mathbf{e}_2) = (0, -2)$, $L(\mathbf{e}_3) = (3, 0)$, where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is the standard basis for $\mathbb{R}^3$.

$$L(x, y, z) = L(x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3)$$
$$= xL(\mathbf{e}_1) + yL(\mathbf{e}_2) + zL(\mathbf{e}_3)$$
$$= x(1, 1) + y(0, -2) + z(3, 0) = (x + 3z, x - 2y)$$

$$L(x, y, z) = \begin{pmatrix} x + 3z \\ x - 2y \end{pmatrix} = \begin{pmatrix} 1 & 0 & 3 \\ 1 & -2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Columns of the matrix are vectors $L(\mathbf{e}_1), L(\mathbf{e}_2), L(\mathbf{e}_3)$.

Theorem Suppose $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map. Then there exists an $m \times n$ matrix A such that $L(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. Columns of A are vectors $L(\mathbf{e}_1), L(\mathbf{e}_2), \dots, L(\mathbf{e}_n)$, where $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ is the standard basis for $\mathbb{R}^n$.

$$\mathbf{y} = A\mathbf{x} \quad \Longleftrightarrow \quad \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

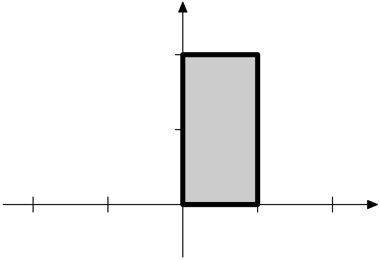
$$\Longleftrightarrow \quad \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \dots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

Linear transformations of $\mathbb{R}^2$

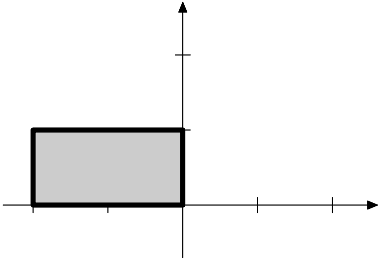
Any linear mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ is represented as multiplication of a 2-dimensional column vector by a 2×2 matrix: $f(\mathbf{x}) = A\mathbf{x}$ or

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

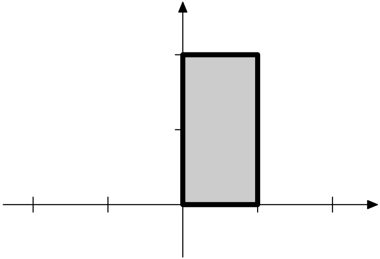
Linear transformations corresponding to particular matrices can have various geometric properties.



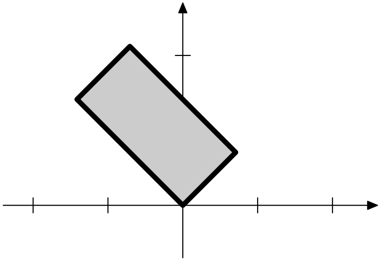
$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$



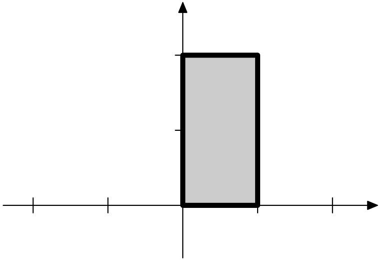
Rotation by 90°



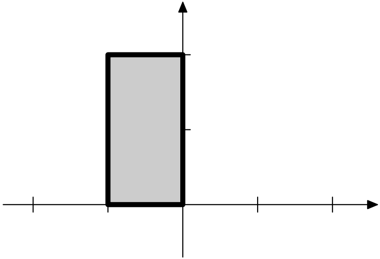
$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$



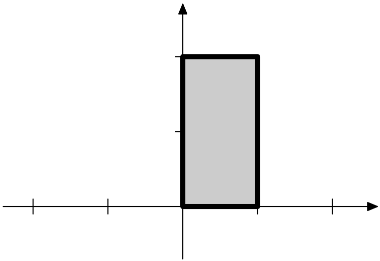
Rotation by 45°



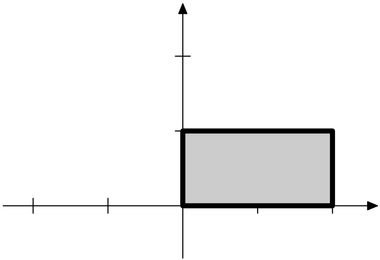
$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$



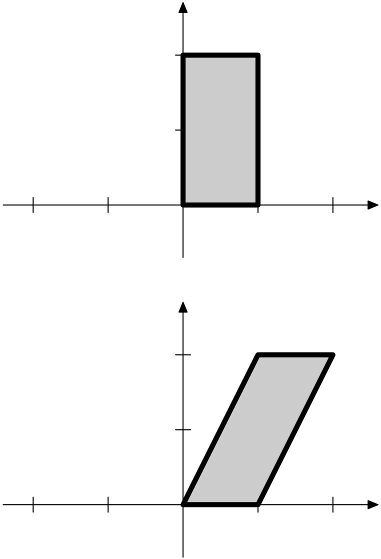
Reflection in
the vertical axis



$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

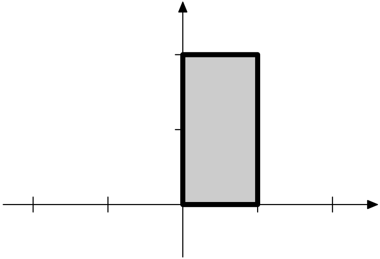


Reflection in
the line $x - y = 0$

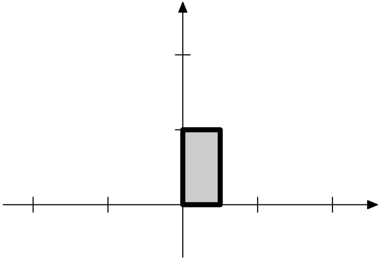


$$A = \begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix}$$

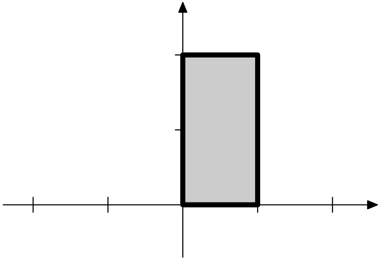
Horizontal shear



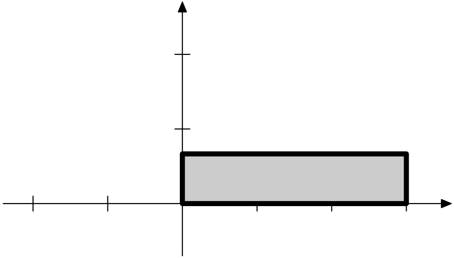
$$A = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$



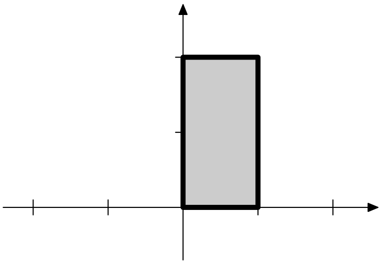
Scaling



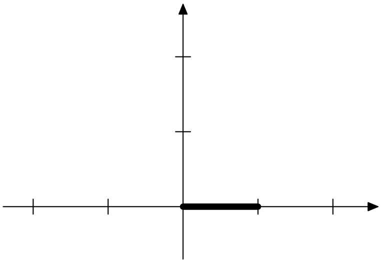
$$A = \begin{pmatrix} 3 & 0 \\ 0 & 1/3 \end{pmatrix}$$



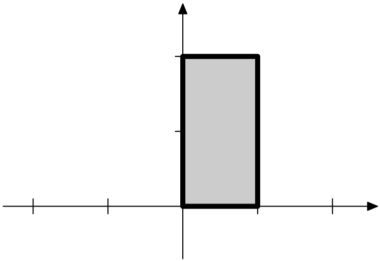
Squeeze



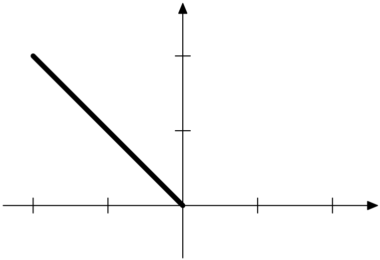
$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$



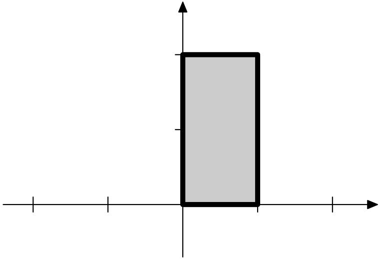
Vertical projection on
the horizontal axis



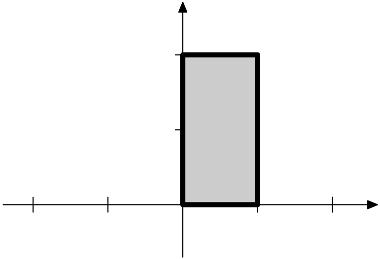
$$A = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}$$



Horizontal projection
on the line $x + y = 0$



$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$



Identity