MATH 304

Linear Algebra

Lecture 28:
Inner product spaces.

Norm

The notion of *norm* generalizes the notion of length of a vector in $\mathbb{R}^n$.

Definition. Let V be a vector space. A function $\alpha : V \to \mathbb{R}$ is called a **norm** on V if it has the following properties:

(i) $\alpha(\mathbf{x}) \geq 0$, $\alpha(\mathbf{x}) = 0$ only for $\mathbf{x} = \mathbf{0}$ (positivity)

(ii) $\alpha(r\mathbf{x}) = |r| \, \alpha(\mathbf{x})$ for all $r \in \mathbb{R}$ (homogeneity)

(iii) $\alpha(\mathbf{x} + \mathbf{y}) \leq \alpha(\mathbf{x}) + \alpha(\mathbf{y})$ (triangle inequality)

Notation. The norm of a vector $\mathbf{x} \in V$ is usually denoted $\|\mathbf{x}\|$. Different norms on V are distinguished by subscripts, e.g., $\|\mathbf{x}\|_1$ and $\|\mathbf{x}\|_2$.

Examples. $V = \mathbb{R}^n$, $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.

- $\|\mathbf{x}\|_\infty = \max(|x_1|, |x_2|, \ldots, |x_n|)$.

- $\|\mathbf{x}\|_p = \left(|x_1|^p + |x_2|^p + \cdots + |x_n|^p\right)^{1/p}$, $p \geq 1$.

In particular, $\|\mathbf{x}\|_2 = |\mathbf{x}|$.

Examples. $V = C[a, b]$, $f : [a, b] \to \mathbb{R}$.

- $\|f\|_\infty = \max\limits_{a \leq x \leq b} |f(x)|$.

- $\|f\|_p = \left(\displaystyle\int_a^b |f(x)|^p \, dx\right)^{1/p}$, $p \geq 1$.

Inner product

The notion of *inner product* generalizes the notion of dot product of vectors in $\mathbb{R}^n$.

Definition. Let V be a vector space. A function $\beta : V \times V \to \mathbb{R}$, usually denoted $\beta(\mathbf{x}, \mathbf{y}) = \langle \mathbf{x}, \mathbf{y} \rangle$, is called an **inner product** on V if it is positive, symmetric, and bilinear. That is, if

(i) $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$, $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ only for $\mathbf{x} = \mathbf{0}$ (positivity)

(ii) $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$ (symmetry)

(iii) $\langle r\mathbf{x}, \mathbf{y} \rangle = r \langle \mathbf{x}, \mathbf{y} \rangle$ (homogeneity)

(iv) $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$ (distributive law)

An **inner product space** is a vector space endowed with an inner product.

Examples. $V = \mathbb{R}^n$.

- $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$.

- $\langle \mathbf{x}, \mathbf{y} \rangle = d_1 x_1 y_1 + d_2 x_2 y_2 + \cdots + d_n x_n y_n$,
where $d_1, d_2, \ldots, d_n > 0$.

Example. $V = \mathcal{P}_n$, polynomials of degree $< n$.

- $\langle p, q \rangle = p(x_1)q(x_1) + p(x_2)q(x_2) + \cdots + p(x_n)q(x_n)$,
where $x_1, x_2, \ldots, x_n$ are distinct points on $\mathbb{R}$.

We have $\langle p, p \rangle = 0 \implies p = 0$ since a nonzero polynomial of degree less than n cannot have n roots.

Examples. $V = C[a, b]$.

- $\langle f, g \rangle = \displaystyle\int_a^b f(x)g(x)\, dx.$

- $\langle f, g \rangle = \displaystyle\int_a^b f(x)g(x)w(x)\, dx,$

where w is bounded, piecewise continuous, and $w > 0$ everywhere on $[a, b]$.

w is called the **weight** function.

Theorem Suppose $\langle \mathbf{x}, \mathbf{y} \rangle$ is an inner product on a vector space V. Then
$$\langle \mathbf{x}, \mathbf{y} \rangle^2 \leq \langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle \quad \text{for all} \ \ \mathbf{x}, \mathbf{y} \in V.$$

Proof: For any $t \in \mathbb{R}$ let $\mathbf{v}_t = \mathbf{x} + t\mathbf{y}$. Then
$$\langle \mathbf{v}_t, \mathbf{v}_t \rangle = \langle \mathbf{x}, \mathbf{x} \rangle + 2t\langle \mathbf{x}, \mathbf{y} \rangle + t^2\langle \mathbf{y}, \mathbf{y} \rangle.$$

The right-hand side is a quadratic polynomial in t (provided that $\mathbf{y} \neq \mathbf{0}$). Since $\langle \mathbf{v}_t, \mathbf{v}_t \rangle \geq 0$ for all t, the discriminant D is nonpositive. But
$D = 4\langle \mathbf{x}, \mathbf{y} \rangle^2 - 4\langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle.$

Cauchy-Schwarz Inequality:
$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} \, \sqrt{\langle \mathbf{y}, \mathbf{y} \rangle}.$$

Cauchy-Schwarz Inequality:
$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} \sqrt{\langle \mathbf{y}, \mathbf{y} \rangle}.$$

Corollary 1 $\quad |\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}| \, |\mathbf{y}|$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

Equivalently, for all $x_i, y_i \in \mathbb{R}$,

$$(x_1 y_1 + \cdots + x_n y_n)^2 \leq (x_1^2 + \cdots + x_n^2)(y_1^2 + \cdots + y_n^2).$$

Corollary 2 For any $f, g \in C[a, b]$,

$$\left(\int_a^b f(x)g(x)\, dx \right)^2 \leq \int_a^b |f(x)|^2 \, dx \cdot \int_a^b |g(x)|^2 \, dx.$$

Norms induced by inner products

Theorem Suppose $\langle \mathbf{x}, \mathbf{y} \rangle$ is an inner product on a vector space V. Then $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$ is a norm.

Proof: Positivity is obvious. Homogeneity:
$$\|r\mathbf{x}\| = \sqrt{\langle r\mathbf{x}, r\mathbf{x} \rangle} = \sqrt{r^2 \langle \mathbf{x}, \mathbf{x} \rangle} = |r| \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}.$$

Triangle inequality (follows from Cauchy-Schwarz's):
$$\|\mathbf{x} + \mathbf{y}\|^2 = \langle \mathbf{x} + \mathbf{y}, \mathbf{x} + \mathbf{y} \rangle$$
$$= \langle \mathbf{x}, \mathbf{x} \rangle + \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{x} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle$$
$$\leq \langle \mathbf{x}, \mathbf{x} \rangle + |\langle \mathbf{x}, \mathbf{y} \rangle| + |\langle \mathbf{y}, \mathbf{x} \rangle| + \langle \mathbf{y}, \mathbf{y} \rangle$$
$$\leq \|\mathbf{x}\|^2 + 2\|\mathbf{x}\| \, \|\mathbf{y}\| + \|\mathbf{y}\|^2 = (\|\mathbf{x}\| + \|\mathbf{y}\|)^2.$$

Examples. • The length of a vector in $\mathbb{R}^n$,
$$|\mathbf{x}| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2},$$
is the norm induced by the dot product
$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

• The norm $\|f\|_2 = \left(\int_a^b |f(x)|^2 \, dx \right)^{1/2}$ on the

vector space $C[a, b]$ is induced by the inner product
$$\langle f, g \rangle = \int_a^b f(x) g(x) \, dx.$$

Angle

Since $|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\| \, \|\mathbf{y}\|$, we can define the *angle* between nonzero vectors in any vector space with an inner product (and induced norm):

$$\angle(\mathbf{x}, \mathbf{y}) = \arccos \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{x}\| \, \|\mathbf{y}\|}.$$

Then $\langle \mathbf{x}, \mathbf{y} \rangle = \|\mathbf{x}\| \, \|\mathbf{y}\| \cos \angle(\mathbf{x}, \mathbf{y})$.

In particular, vectors $\mathbf{x}$ and $\mathbf{y}$ are **orthogonal** (denoted $\mathbf{x} \perp \mathbf{y}$) if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$.

Problem. Find the angle between functions $f_1(x) = x$ and $f_2(x) = x^2$ in the inner space $C[0, 1]$ with the inner product
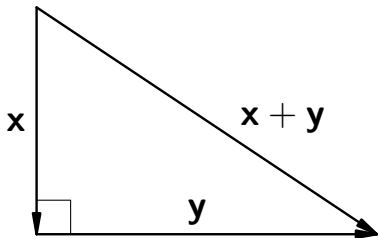
$$\langle f, g \rangle = \int_0^1 f(x)g(x)\, dx.$$

$$\langle f_1, f_2 \rangle = \int_0^1 x \cdot x^2 \, dx = \frac{1}{4},$$

$$\langle f_1, f_1 \rangle = \int_0^1 x^2 \, dx = \frac{1}{3}, \quad \langle f_2, f_2 \rangle = \int_0^1 (x^2)^2 \, dx = \frac{1}{5}.$$

$$\cos \angle(f_1, f_2) = \frac{\langle f_1, f_2 \rangle}{\|f_1\| \, \|f_2\|} = \frac{1/4}{\sqrt{1/3}\sqrt{1/5}} = \frac{\sqrt{15}}{4}$$

$$\sin \angle(f_1, f_2) = \sqrt{1 - \cos^2 \angle(f_1, f_2)} = 1/4$$
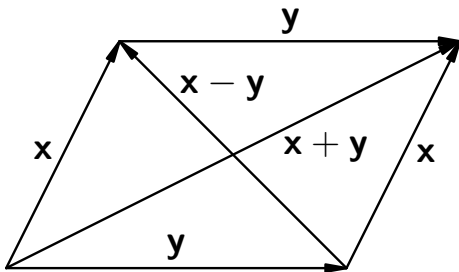
$$\angle(f_1, f_2) = \arcsin \frac{1}{4}$$

Pythagorean Theorem:

$$\mathbf{x} \perp \mathbf{y} \implies \|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$$

Proof:
$$\|\mathbf{x} + \mathbf{y}\|^2 = \langle \mathbf{x} + \mathbf{y}, \mathbf{x} + \mathbf{y} \rangle$$
$$= \langle \mathbf{x}, \mathbf{x} \rangle + \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{x} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle$$
$$= \langle \mathbf{x}, \mathbf{x} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2.$$

Parallelogram Identity:
$$\|\mathbf{x} + \mathbf{y}\|^2 + \|\mathbf{x} - \mathbf{y}\|^2 = 2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2$$

Proof: $\|\mathbf{x}+\mathbf{y}\|^2 = \langle \mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{x} \rangle + \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{x} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle$.
Similarly, $\|\mathbf{x}-\mathbf{y}\|^2 = \langle \mathbf{x}, \mathbf{x} \rangle - \langle \mathbf{x}, \mathbf{y} \rangle - \langle \mathbf{y}, \mathbf{x} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle$.
Then $\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2 = 2\langle \mathbf{x}, \mathbf{x} \rangle + 2\langle \mathbf{y}, \mathbf{y} \rangle = 2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2$.

Example. Norms on $\mathbb{R}^n$, $n \geq 2$:

- $\|\mathbf{x}\|_\infty = \max(|x_1|, |x_2|, \ldots, |x_n|)$,
- $\|\mathbf{x}\|_p = \left(|x_1|^p + |x_2|^p + \cdots + |x_n|^p\right)^{1/p}$, $\quad p \geq 1$.

Theorem The norms $\|\mathbf{x}\|_\infty$ and $\|\mathbf{x}\|_p$, $p \neq 2$ do not satisfy the Parallelogram Identity. Hence they are not induced by any inner product on $\mathbb{R}^n$.

Proof: A counterexample to the Parallelogram Identity is provided by vectors $\mathbf{e}_1 = (1, 0, 0, \ldots, 0)$ and $\mathbf{e}_2 = (0, 1, 0, \ldots, 0)$.

$\|\mathbf{e}_1\|_\infty = \|\mathbf{e}_2\|_\infty = 1$, $\|\mathbf{e}_1\|_p = \|\mathbf{e}_2\|_p = 1$ for any $p \geq 1$.

$\|\mathbf{e}_1 + \mathbf{e}_2\|_\infty = \|\mathbf{e}_1 - \mathbf{e}_2\|_\infty = 1$,

$\|\mathbf{e}_1 + \mathbf{e}_2\|_p = \|\mathbf{e}_1 - \mathbf{e}_2\|_p = 2^{1/p}$ for any $p \geq 1$.

Thus $2\|\mathbf{e}_1\|_\infty^2 + 2\|\mathbf{e}_2\|_\infty^2 = 2\|\mathbf{e}_1\|_p^2 + 2\|\mathbf{e}_2\|_p^2 = 4$.

On the other hand, $\|\mathbf{e}_1 + \mathbf{e}_2\|_\infty^2 + \|\mathbf{e}_1 - \mathbf{e}_2\|_\infty^2 = 2 \neq 4$ and

$\|\mathbf{e}_1 + \mathbf{e}_2\|_p^2 + \|\mathbf{e}_1 - \mathbf{e}_2\|_p^2 = 2(2^{1/p})^2 = 2^{1+2/p} \neq 4$ unless $p = 2$.