MATH 304

Linear algebra

Lecture 35:
Matrix exponentials.

- *Initial value problem for a linear ODE:*

$$\frac{dy}{dt} = 2y, \quad y(0) = 3.$$

Solution: $y(t) = 3e^{2t}$.

- *Initial value problem for a system of linear ODEs:*

$$\begin{cases} \frac{dx}{dt} = 2x + 3y, \\ \frac{dy}{dt} = x + 4y, \end{cases} \qquad x(0) = 2, \ \ y(0) = 1.$$

The system can be rewritten in vector form

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}, \ \text{where} \ \ A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}.$$

Solution: $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{tA}\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$

What is e^{tA}*?*

Exponential function

Exponential function: $f(x) = \exp x = e^x, \; x \in \mathbb{R}$.

We have that $e^{x+y} = e^x \cdot e^y$.

Definition 1. $e^x = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n$.

In particular, $e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n \approx 2.7182818$.

Definition 2. $e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$

Definition 3. $f(x) = e^x$ is the unique solution of the initial value problem $f' = f, \; f(0) = 1$.

Matrix exponentials

Definition. For any square matrix A let

$$\exp A = e^A = I + A + \frac{1}{2!}A^2 + \cdots + \frac{1}{n!}A^n + \cdots$$

Matrix exponential is a limit of matrix polynomials.

Remark. Let $A^{(1)}, A^{(2)}, \ldots$ be a sequence of $n \times n$ matrices, $A^{(n)} = (a_{ij}^{(n)})$. The sequence converges to an $n \times n$ matrix $B = (b_{ij})$ if $a_{ij}^{(n)} \to b_{ij}$ as $n \to \infty$, i.e., if each entry converges.

Theorem The matrix $\exp A$ is well defined, i.e., the series converges.

Properties of matrix exponentials

Theorem 1 If $AB = BA$ then $e^A e^B = e^B e^A = e^{A+B}$.

Corollary (a) $e^{tA} e^{sA} = e^{sA} e^{tA} = e^{(t+s)A}$, $t, s \in \mathbb{R}$;
(b) $e^O = I$; **(c)** $(e^A)^{-1} = e^{-A}$.

Theorem 2 $\frac{d}{dt} e^{tA} = Ae^{tA} = e^{tA}A$.

Indeed, $e^{tA} = I + tA + \frac{t^2}{2!}A^2 + \cdots + \frac{t^n}{n!}A^n + \cdots$,

and the series can be differentiated term by term.

$$\frac{d}{dt}\left(\frac{t^n}{n!}A^n\right) = \frac{d}{dt}\left(\frac{t^n}{n!}\right) A^n = \frac{t^{n-1}}{(n-1)!}A^n.$$

Lemma Let A be an $n \times n$ matrix and $\mathbf{x} \in \mathbb{R}^n$. Then the vector function $\mathbf{v}(t) = e^{tA}\mathbf{x}$ satisfies $\mathbf{v}' = A\mathbf{v}$.

Proof: $\dfrac{d\mathbf{v}}{dt} = \dfrac{d}{dt} e^{tA}\mathbf{x} = \left(Ae^{tA}\right)\mathbf{x} = A\left(e^{tA}\mathbf{x}\right) = A\mathbf{v}$.

Theorem For any $t_0 \in \mathbb{R}$ and $\mathbf{x}_0 \in \mathbb{R}^n$ the initial value problem

$$\frac{d\mathbf{v}}{dt} = A\mathbf{v}, \quad \mathbf{v}(t_0) = \mathbf{x}_0$$

has a unique solution $\mathbf{v}(t) = e^{(t-t_0)A}\,\mathbf{x}_0$.

Indeed, $\mathbf{v}(t) = e^{(t-t_0)A}\,\mathbf{x}_0 = e^{tA}e^{-t_0A}\mathbf{x}_0 = e^{tA}\mathbf{x}$, where $\mathbf{x} = e^{-t_0A}\mathbf{x}_0$ is a constant vector.

Evaluation of matrix exponentials

Example. $A = \mathrm{diag}(a_1, a_2, \ldots, a_k)$.

$$A^n = \mathrm{diag}(a_1^n, a_2^n, \ldots, a_k^n), \quad n = 1, 2, 3, \ldots$$

$$e^A = I + A + \frac{1}{2!}A^2 + \cdots + \frac{1}{n!}A^n + \cdots$$

$$= \mathrm{diag}(b_1, b_2, \ldots, b_k),$$

where $b_i = 1 + a_i + \frac{1}{2!}a_i^2 + \frac{1}{3!}a_i^3 + \cdots = e^{a_i}$.

Theorem If $A = \mathrm{diag}(a_1, a_2, \ldots, a_k)$ then

$$e^A = \mathrm{diag}(e^{a_1}, e^{a_2}, \ldots, e^{a_k}),$$

$$e^{tA} = \mathrm{diag}(e^{a_1 t}, e^{a_2 t}, \ldots, e^{a_k t}).$$

Let A be an n-by-n matrix and suppose there exists a basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$ for $\mathbb{R}^n$ consisting of eigenvectors of A. That is, $A\mathbf{v}_k = \lambda_k \mathbf{v}_k$, where $\lambda_k \in \mathbb{R}$.

Then $A = UBU^{-1}$, where $B = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and U is a transition matrix whose columns are vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

$A^2 = UBU^{-1}UBU^{-1} = UB^2U^{-1}$,
$A^3 = A^2A = UB^2U^{-1}UBU^{-1} = UB^3U^{-1}$.
Likewise, $A^n = UB^nU^{-1}$ for any $n \geq 1$.

$I + 2A - 3A^2 = UIU^{-1} + 2UBU^{-1} - 3UB^2U^{-1} =$
$= U(I + 2B - 3B^2)U^{-1}$.

$\Longrightarrow$ $p(A) = Up(B)U^{-1}$ for any polynomial $p(x)$.
$\Longrightarrow$ $e^{tA} = Ue^{tB}U^{-1}$ for all $t \in \mathbb{R}$.

Example. $A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$.

The eigenvalues of A: $\lambda_1 = 1$, $\lambda_2 = 5$.
Eigenvectors: $\mathbf{v}_1 = (3, -1)$, $\mathbf{v}_2 = (1, 1)$.

Therefore $A = UBU^{-1}$, where

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}, \quad U = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}.$$

Then

$$e^{tA} = Ue^{tB}U^{-1} = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & e^{5t} \end{pmatrix} \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 3e^t & e^{5t} \\ -e^t & e^{5t} \end{pmatrix} \tfrac{1}{4} \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} = \tfrac{1}{4} \begin{pmatrix} 3e^t + e^{5t} & -3e^t + 3e^{5t} \\ -e^t + e^{5t} & e^t + 3e^{5t} \end{pmatrix}.$$

Problem. Solve a system of differential equations

$$\begin{cases} \frac{dx}{dt} = 2x + 3y, \\ \frac{dy}{dt} = x + 4y \end{cases}$$

subject to initial conditions $x(0) = 2,\ y(0) = 1$.

The unique solution:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{tA}\mathbf{x}_0, \quad \text{where} \quad A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}, \ \ \mathbf{x}_0 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

$$e^{tA} = \frac{1}{4}\begin{pmatrix} 3e^t + e^{5t} & -3e^t + 3e^{5t} \\ -e^t + e^{5t} & e^t + 3e^{5t} \end{pmatrix}$$

$$\Longrightarrow \begin{cases} x(t) = \frac{3}{4}e^t + \frac{5}{4}e^{5t}, \\ y(t) = -\frac{1}{4}e^t + \frac{5}{4}e^{5t}. \end{cases}$$

Example. $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, a Jordan block.

$$A^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A^3 = O, \quad A^n = O \text{ for } n \geq 3.$$

$$e^A = I + A + \frac{1}{2}A^2 = \begin{pmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$e^{tA} = I + tA + \frac{t^2}{2}A^2 = \begin{pmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}.$$

Example. $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, another Jordan block.

$$A^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \ A^3 = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \ A^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$

$$e^{tA} = I + tA + \frac{t^2}{2!}A^2 + \cdots + \frac{t^n}{n!}A^n + \cdots = \begin{pmatrix} a(t) & b(t) \\ 0 & a(t) \end{pmatrix},$$

where $a(t) = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots = e^t$,
$b(t) = t + 2\frac{t^2}{2!} + 3\frac{t^3}{3!} + \cdots = te^t$.

Thus $e^{tA} = \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix}$.

Example. $A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, a general Jordan block.

We have that $A = \lambda I + B$, where $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Since $(\lambda I)B = B(\lambda I)$, it follows that $e^A = e^{\lambda I} e^B$.
Similarly, $e^{tA} = e^{t\lambda I} e^{tB}$.

$$e^{t\lambda I} = \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{\lambda t} \end{pmatrix} = e^{\lambda t} I,$$

$$B^2 = O \implies e^{tB} = I + tB = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

Thus $e^{tA} = e^{t\lambda I} e^{tB} = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}$.

Problem. Solve a system of differential equations

$$\begin{cases} \frac{dx}{dt} = 2x + y, \\ \frac{dy}{dt} = 2y \end{cases}$$

subject to initial conditions $x(0) = y(0) = 1$.

The unique solution:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{tA}\mathbf{x}_0, \text{ where } A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \ \mathbf{x}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$$e^{tA} = \begin{pmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{pmatrix} = e^{2t}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

$$\Longrightarrow \begin{cases} x(t) = e^{2t}(1+t), \\ y(t) = e^{2t}. \end{cases}$$