

MATH 304

Linear algebra

Lecture 37:
Rotations in space.

Orthogonal matrices

Definition. An $n \times n$ matrix A is called **orthogonal** if $AA^T = A^T A = I$, i.e., $A^T = A^{-1}$.

Example. $A = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix}$.

- A_ϕ is orthogonal.
- Eigenvalues: $\lambda_1 = \cos \phi + i \sin \phi = e^{i\phi}$,
 $\lambda_2 = \cos \phi - i \sin \phi = e^{-i\phi}$.
- $|\lambda_1| = |\lambda_2| = 1$, $\overline{\lambda_1} = \lambda_2$.
- Associated eigenvectors: $\mathbf{v}_1 = (1, -i)$,
 $\mathbf{v}_2 = (1, i)$.
- Vectors \mathbf{v}_1 and \mathbf{v}_2 form a basis for \mathbb{C}^2 .

Consider a linear operator $L : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $L(\mathbf{x}) = A\mathbf{x}$, where A is an $n \times n$ orthogonal matrix.

Theorem There exists an orthonormal basis for \mathbb{R}^n such that the matrix of L relative to this basis has a diagonal block structure

$$\begin{pmatrix} D_{\pm 1} & O & \dots & O \\ O & R_1 & \dots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \dots & R_k \end{pmatrix},$$

where $D_{\pm 1}$ is a diagonal matrix whose diagonal entries are equal to 1 or -1 , and

$$R_j = \begin{pmatrix} \cos \phi_j & -\sin \phi_j \\ \sin \phi_j & \cos \phi_j \end{pmatrix}, \quad \phi_j \in \mathbb{R}.$$

Why are orthogonal matrices called so?

Theorem Given an $n \times n$ matrix A , the following conditions are equivalent:

- (i) A is orthogonal: $A^T = A^{-1}$;
- (ii) columns of A form an orthonormal basis for \mathbb{R}^n ;
- (iii) rows of A form an orthonormal basis for \mathbb{R}^n .

Proof: Entries of the matrix $A^T A$ are dot products of columns of A . Entries of AA^T are dot products of rows of A .

Thus an orthogonal matrix is the transition matrix from one orthonormal basis to another.

Consider a linear operator $L : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $L(\mathbf{x}) = A\mathbf{x}$, where A is an $n \times n$ matrix.

Theorem The following conditions are equivalent:

- (i) $|L(\mathbf{x})| = |\mathbf{x}|$ for all $\mathbf{x} \in \mathbb{R}^n$;
- (ii) $L(\mathbf{x}) \cdot L(\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$;
- (iii) the matrix A is orthogonal.

Definition. A transformation $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is called an **isometry** if it preserves distances between points: $|f(\mathbf{x}) - f(\mathbf{y})| = |\mathbf{x} - \mathbf{y}|$.

Theorem Any isometry $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ can be represented as $f(\mathbf{x}) = A\mathbf{x} + \mathbf{x}_0$, where $\mathbf{x}_0 \in \mathbb{R}^n$ and A is an orthogonal matrix.

Classification of 3×3 orthogonal matrices:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi & \cos \phi \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

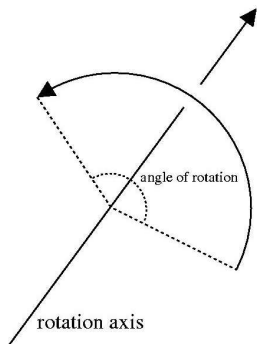
$$C = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi & \cos \phi \end{pmatrix}.$$

A = rotation about a line; B = reflection in a plane; C = rotation about a line combined with reflection in the orthogonal plane.

$$\det A = 1, \quad \det B = \det C = -1.$$

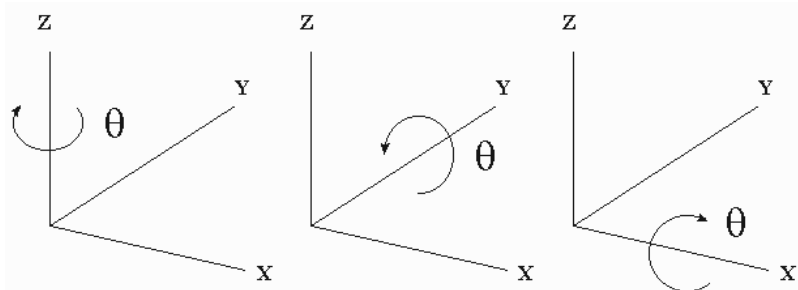
A has eigenvalues $1, e^{i\phi}, e^{-i\phi}$. B has eigenvalues $-1, 1, 1$. C has eigenvalues $-1, e^{i\phi}, e^{-i\phi}$.

Rotations in space



If the axis of rotation is oriented, we can say about *clockwise* or *counterclockwise* rotations (with respect to the view from the positive semi-axis).

Clockwise rotations about coordinate axes



$$\begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{pmatrix}$$

Problem. Find the matrix of the rotation by 90° about the line spanned by the vector $\mathbf{c} = (1, 2, 2)$. The rotation is assumed to be counterclockwise when looking from the tip of \mathbf{c} .

$B = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is the matrix of (counterclockwise) rotation by 90° about the z-axis.

We need to find an orthonormal basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ such that \mathbf{v}_3 has the same direction as \mathbf{c} . Also, the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ should obey the same hand rule as the standard basis. Then B is the matrix of the given rotation relative to the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

Let U denote the transition matrix from the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to the standard basis (columns of U are vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$). Then the desired matrix is $A = UBU^{-1}$.

Since $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is going to be an orthonormal basis, the matrix U will be orthogonal. Then $U^{-1} = U^T$ and $A = UBU^T$.

Remark. The basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ obeys the same hand rule as the standard basis if and only if $\det U > 0$.

Hint. Vectors $\mathbf{a} = (-2, -1, 2)$, $\mathbf{b} = (2, -2, 1)$, and $\mathbf{c} = (1, 2, 2)$ are orthogonal.

We have $|\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}| = 3$, hence $\mathbf{v}_1 = \frac{1}{3}\mathbf{a}$, $\mathbf{v}_2 = \frac{1}{3}\mathbf{b}$, $\mathbf{v}_3 = \frac{1}{3}\mathbf{c}$ is an orthonormal basis.

Transition matrix: $U = \frac{1}{3} \begin{pmatrix} -2 & 2 & 1 \\ -1 & -2 & 2 \\ 2 & 1 & 2 \end{pmatrix}$.

$$\det U = \frac{1}{27} \begin{vmatrix} -2 & 2 & 1 \\ -1 & -2 & 2 \\ 2 & 1 & 2 \end{vmatrix} = \frac{1}{27} \cdot 27 = 1.$$

(In the case $\det U = -1$, we should interchange vectors \mathbf{v}_1 and \mathbf{v}_2 .)

$$A = UBU^T$$

$$= \frac{1}{3} \begin{pmatrix} -2 & 2 & 1 \\ -1 & -2 & 2 \\ 2 & 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \frac{1}{3} \begin{pmatrix} -2 & -1 & 2 \\ 2 & -2 & 1 \\ 1 & 2 & 2 \end{pmatrix}$$

$$= \frac{1}{9} \begin{pmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{pmatrix} \begin{pmatrix} -2 & -1 & 2 \\ 2 & -2 & 1 \\ 1 & 2 & 2 \end{pmatrix}$$

$$= \frac{1}{9} \begin{pmatrix} 1 & -4 & 8 \\ 8 & 4 & 1 \\ -4 & 7 & 4 \end{pmatrix}.$$

$U = \frac{1}{3} \begin{pmatrix} -2 & 2 & 1 \\ -1 & -2 & 2 \\ 2 & 1 & 2 \end{pmatrix}$ is an orthogonal matrix.

$\det U = 1 \implies U$ is a rotation matrix.

Problem. (a) Find the axis of the rotation.

(b) Find the angle of the rotation.

The axis is the set of points $\mathbf{x} \in \mathbb{R}^n$ such that $U\mathbf{x} = \mathbf{x} \iff (U - I)\mathbf{x} = \mathbf{0}$. To find the axis, we apply row reduction to the matrix $3(U - I)$:

$$3U - 3I = \begin{pmatrix} -5 & 2 & 1 \\ -1 & -5 & 2 \\ 2 & 1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} -3 & 3 & 0 \\ -1 & -5 & 2 \\ 2 & 1 & -1 \end{pmatrix}$$

$$\begin{aligned} &\rightarrow \begin{pmatrix} 1 & -1 & 0 \\ -1 & -5 & 2 \\ 2 & 1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 0 \\ 0 & -6 & 2 \\ 2 & 1 & -1 \end{pmatrix} \rightarrow \\ &\begin{pmatrix} 1 & -1 & 0 \\ 0 & -6 & 2 \\ 0 & 3 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 0 \\ 0 & 3 & -1 \\ 0 & 0 & 0 \end{pmatrix} \\ &\rightarrow \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1/3 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1/3 \\ 0 & 1 & -1/3 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

$$\text{Thus } U\mathbf{x} = \mathbf{x} \iff \begin{cases} x - z/3 = 0 \\ y - z/3 = 0 \end{cases}$$

The general solution is $x = y = t/3$, $z = t$, $t \in \mathbb{R}$.

$\implies \mathbf{d} = (1, 1, 3)$ is the direction of the axis.

$$U = \frac{1}{3} \begin{pmatrix} -2 & 2 & 1 \\ -1 & -2 & 2 \\ 2 & 1 & 2 \end{pmatrix}$$

Let ϕ be the angle of rotation. Then the eigenvalues of U are 1, $e^{i\phi}$, and $e^{-i\phi}$. Therefore

$$\det(U - \lambda I) = (1 - \lambda)(e^{i\phi} - \lambda)(e^{-i\phi} - \lambda).$$

Besides, $\det(U - \lambda I) = -\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3$, where $c_1 = \text{tr } U$ (the sum of diagonal entries).

It follows that

$$\text{tr } U = 1 + e^{i\phi} + e^{-i\phi} = 1 + 2 \cos \phi.$$

$$\text{tr } U = -2/3 \implies \cos \phi = -5/6 \implies \phi \approx 146.44^\circ$$