

MATH 304  
Linear Algebra

**Lecture 4:**  
**Matrix multiplication.**  
**Diagonal matrices.**  
**Inverse matrix.**

## Matrices

*Definition.* An **m-by-n matrix** is a rectangular array of numbers that has  $m$  rows and  $n$  columns:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

*Notation:*  $A = (a_{ij})_{1 \leq i \leq n, 1 \leq j \leq m}$  or simply  $A = (a_{ij})$  if the dimensions are known.

## Matrix algebra: linear operations

**Addition:** two matrices of the same dimensions can be added by adding their corresponding entries.

**Scalar multiplication:** to multiply a matrix  $A$  by a scalar  $r$ , one multiplies each entry of  $A$  by  $r$ .

**Zero matrix  $O$ :** all entries are zeros.

**Negative:**  $-A$  is defined as  $(-1)A$ .

**Subtraction:**  $A - B$  is defined as  $A + (-B)$ .

As far as the linear operations are concerned, the  $m \times n$  matrices can be regarded as  $mn$ -dimensional vectors.

## Properties of linear operations

$$(A + B) + C = A + (B + C)$$

$$A + B = B + A$$

$$A + O = O + A = A$$

$$A + (-A) = (-A) + A = O$$

$$r(sA) = (rs)A$$

$$r(A + B) = rA + rB$$

$$(r + s)A = rA + sA$$

$$1A = A$$

$$0A = O$$

## Dot product

*Definition.* The **dot product** of  $n$ -dimensional vectors  $\mathbf{x} = (x_1, x_2, \dots, x_n)$  and  $\mathbf{y} = (y_1, y_2, \dots, y_n)$  is a scalar

$$\mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2 + \cdots + x_ny_n = \sum_{k=1}^n x_ky_k.$$

The dot product is also called the **scalar product**.

## Matrix multiplication

The product of matrices  $A$  and  $B$  is defined if the number of columns in  $A$  matches the number of rows in  $B$ .

*Definition.* Let  $A = (a_{ik})$  be an  $m \times n$  matrix and  $B = (b_{kj})$  be an  $n \times p$  matrix. The **product**  $AB$  is defined to be the  $m \times p$  matrix  $C = (c_{ij})$  such that

$$c_{ij} = \sum_{k=1}^n a_{ik} b_{kj} \text{ for all indices } i, j.$$

That is, matrices are multiplied **row by column**:

$$\begin{pmatrix} * & * & * \\ \boxed{*} & \boxed{*} & \boxed{*} \end{pmatrix} \begin{pmatrix} * & * & \boxed{*} & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix} = \begin{pmatrix} * & * & * & * \\ * & * & \boxed{*} & * \end{pmatrix}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} = \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_m \end{pmatrix}$$

$$B = \left( \begin{array}{c|c|c|c} b_{11} & b_{12} & \dots & b_{1p} \\ b_{21} & b_{22} & \dots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{np} \end{array} \right) = (\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_p)$$

$$\Rightarrow AB = \begin{pmatrix} \mathbf{v}_1 \cdot \mathbf{w}_1 & \mathbf{v}_1 \cdot \mathbf{w}_2 & \dots & \mathbf{v}_1 \cdot \mathbf{w}_p \\ \mathbf{v}_2 \cdot \mathbf{w}_1 & \mathbf{v}_2 \cdot \mathbf{w}_2 & \dots & \mathbf{v}_2 \cdot \mathbf{w}_p \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{v}_m \cdot \mathbf{w}_1 & \mathbf{v}_m \cdot \mathbf{w}_2 & \dots & \mathbf{v}_m \cdot \mathbf{w}_p \end{pmatrix}$$

*Examples.*

$$(x_1, x_2, \dots, x_n) \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \left( \sum_{k=1}^n x_k y_k \right),$$

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} (x_1, x_2, \dots, x_n) = \begin{pmatrix} y_1 x_1 & y_1 x_2 & \dots & y_1 x_n \\ y_2 x_1 & y_2 x_2 & \dots & y_2 x_n \\ \vdots & \vdots & \ddots & \vdots \\ y_n x_1 & y_n x_2 & \dots & y_n x_n \end{pmatrix}.$$



*Example.*

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 3 & 1 & 1 \\ -2 & 5 & 6 & 0 \\ 1 & 7 & 4 & 1 \end{pmatrix} = \begin{pmatrix} -3 & 1 & 3 & 0 \\ -3 & 17 & 16 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 3 & 1 & 1 \\ -2 & 5 & 6 & 0 \\ 1 & 7 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 1 \end{pmatrix} \text{ is not defined}$$

System of linear equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

Matrix representation of the system:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \iff \mathbf{Ax} = \mathbf{b},$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

## Properties of matrix multiplication:

$$(AB)C = A(BC) \quad (\text{associative law})$$

$$(A + B)C = AC + BC \quad (\text{distributive law \#1})$$

$$C(A + B) = CA + CB \quad (\text{distributive law \#2})$$

$$(rA)B = A(rB) = r(AB)$$

*Any of the above identities holds provided that matrix sums and products are well defined.*

If  $A$  and  $B$  are  $n \times n$  matrices, then both  $AB$  and  $BA$  are well defined  $n \times n$  matrices.

However, in general,  $AB \neq BA$ .

*Example.* Let  $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ ,  $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ .

Then  $AB = \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$ ,  $BA = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$ .

If  $AB$  does equal  $BA$ , we say that the matrices  $A$  and  $B$  **commute**.

**Problem.** Let  $A$  and  $B$  be arbitrary  $n \times n$  matrices. Is it true that  $(A - B)(A + B) = A^2 - B^2$ ?

$$\begin{aligned}(A - B)(A + B) &= (A - B)A + (A - B)B \\ &= (AA - BA) + (AB - BB) \\ &= A^2 + AB - BA - B^2\end{aligned}$$

Hence  $(A - B)(A + B) = A^2 - B^2$  if and only if  $A$  commutes with  $B$ .

## Diagonal matrices

If  $A = (a_{ij})$  is a square matrix, then the entries  $a_{ij}$  are called **diagonal entries**. A square matrix is called **diagonal** if all non-diagonal entries are zeros.

*Example.*  $\begin{pmatrix} 7 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ , denoted  $\text{diag}(7, 1, 2)$ .

Let  $A = \text{diag}(s_1, s_2, \dots, s_n)$ ,  $B = \text{diag}(t_1, t_2, \dots, t_n)$ .

Then  $A + B = \text{diag}(s_1 + t_1, s_2 + t_2, \dots, s_n + t_n)$ ,

$$rA = \text{diag}(rs_1, rs_2, \dots, rs_n).$$

*Example.*

$$\begin{pmatrix} 7 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} -7 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

**Theorem** Let  $A = \text{diag}(s_1, s_2, \dots, s_n)$ ,

$B = \text{diag}(t_1, t_2, \dots, t_n)$ .

Then  $A + B = \text{diag}(s_1 + t_1, s_2 + t_2, \dots, s_n + t_n)$ ,

$rA = \text{diag}(rs_1, rs_2, \dots, rs_n)$ .

$AB = \text{diag}(s_1 t_1, s_2 t_2, \dots, s_n t_n)$ .

In particular, diagonal matrices always commute.



*Example.*

$$\begin{pmatrix} 7 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 7a_{11} & 7a_{12} & 7a_{13} \\ a_{21} & a_{22} & a_{23} \\ 2a_{31} & 2a_{32} & 2a_{33} \end{pmatrix}$$

**Theorem** Let  $D = \text{diag}(d_1, d_2, \dots, d_m)$  and  $A$  be an  $m \times n$  matrix. Then the matrix  $DA$  is obtained from  $A$  by multiplying the  $i$ th row by  $d_i$  for  $i = 1, 2, \dots, m$ :

$$A = \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_m \end{pmatrix} \implies DA = \begin{pmatrix} d_1 \mathbf{v}_1 \\ d_2 \mathbf{v}_2 \\ \vdots \\ d_m \mathbf{v}_m \end{pmatrix}$$

*Example.*

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} 7 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 7a_{11} & a_{12} & 2a_{13} \\ 7a_{21} & a_{22} & 2a_{23} \\ 7a_{31} & a_{32} & 2a_{33} \end{pmatrix}$$

**Theorem** Let  $D = \text{diag}(d_1, d_2, \dots, d_n)$  and  $A$  be an  $m \times n$  matrix. Then the matrix  $AD$  is obtained from  $A$  by multiplying the  $i$ th column by  $d_i$  for  $i = 1, 2, \dots, n$ :

$$\begin{aligned} A &= (\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n) \\ \implies AD &= (d_1\mathbf{w}_1, d_2\mathbf{w}_2, \dots, d_n\mathbf{w}_n) \end{aligned}$$

## Identity matrix

*Definition.* The **identity matrix** (or **unit matrix**) is a diagonal matrix with all diagonal entries equal to 1. The  $n \times n$  identity matrix is denoted  $I_n$  or simply  $I$ .

$$I_1 = (1), \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In general, 
$$I = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}.$$

**Theorem.** Let  $A$  be an arbitrary  $m \times n$  matrix. Then  $I_m A = A I_n = A$ .

## Inverse matrix

Let  $\mathcal{M}_n(\mathbb{R})$  denote the set of all  $n \times n$  matrices with real entries. We can **add**, **subtract**, and **multiply** elements of  $\mathcal{M}_n(\mathbb{R})$ . What about **division**?

*Definition.* Let  $A \in \mathcal{M}_n(\mathbb{R})$ . Suppose there exists an  $n \times n$  matrix  $B$  such that

$$AB = BA = I_n.$$

Then the matrix  $A$  is called **invertible** and  $B$  is called the **inverse** of  $A$  (denoted  $A^{-1}$ ).

A non-invertible square matrix is called **singular**.

$$\boxed{AA^{-1} = A^{-1}A = I}$$

## Examples

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

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$$AB = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$BA = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$C^2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus  $A^{-1} = B$ ,  $B^{-1} = A$ , and  $C^{-1} = C$ .