

MATH 304  
Linear Algebra

**Lecture 9:**  
**Subspaces of vector spaces (continued).**  
**Span. Spanning set.**

## Vector space

A *vector space* is a set  $V$  equipped with two operations, **addition**

$$V \times V \ni (\mathbf{x}, \mathbf{y}) \mapsto \mathbf{x} + \mathbf{y} \in V$$

and **scalar multiplication**

$$\mathbb{R} \times V \ni (r, \mathbf{x}) \mapsto r\mathbf{x} \in V,$$

that have the following properties:

## Properties of addition and scalar multiplication

**A1.**  $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$  for all  $\mathbf{a}, \mathbf{b} \in V$ .

**A2.**  $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$  for all  $\mathbf{a}, \mathbf{b}, \mathbf{c} \in V$ .

**A3.** There exists an element of  $V$ , called the *zero vector* and denoted  $\mathbf{0}$ , such that  $\mathbf{a} + \mathbf{0} = \mathbf{0} + \mathbf{a} = \mathbf{a}$  for all  $\mathbf{a} \in V$ .

**A4.** For any  $\mathbf{a} \in V$  there exists an element of  $V$ , denoted  $-\mathbf{a}$ , such that  $\mathbf{a} + (-\mathbf{a}) = (-\mathbf{a}) + \mathbf{a} = \mathbf{0}$ .

**A5.**  $r(\mathbf{a} + \mathbf{b}) = r\mathbf{a} + r\mathbf{b}$  for all  $r \in \mathbb{R}$  and  $\mathbf{a}, \mathbf{b} \in V$ .

**A6.**  $(r + s)\mathbf{a} = r\mathbf{a} + s\mathbf{a}$  for all  $r, s \in \mathbb{R}$  and  $\mathbf{a} \in V$ .

**A7.**  $(rs)\mathbf{a} = r(s\mathbf{a})$  for all  $r, s \in \mathbb{R}$  and  $\mathbf{a} \in V$ .

**A8.**  $1\mathbf{a} = \mathbf{a}$  for all  $\mathbf{a} \in V$ .

## Examples of vector spaces

- $\mathbb{R}^n$ :  $n$ -dimensional coordinate vectors
- $\mathcal{M}_{m,n}(\mathbb{R})$ :  $m \times n$  matrices with real entries
- $\mathbb{R}^\infty$ : infinite sequences  $(x_1, x_2, \dots)$ ,  $x_i \in \mathbb{R}$
- $\{\mathbf{0}\}$ : the trivial vector space
- $F(\mathbb{R})$ : the set of all functions  $f : \mathbb{R} \rightarrow \mathbb{R}$
- $C(\mathbb{R})$ : all continuous functions  $f : \mathbb{R} \rightarrow \mathbb{R}$
- $C^1(\mathbb{R})$ : all continuously differentiable functions  $f : \mathbb{R} \rightarrow \mathbb{R}$
- $C^\infty(\mathbb{R})$ : all smooth functions  $f : \mathbb{R} \rightarrow \mathbb{R}$
- $\mathcal{P}$ : all polynomials  $p(x) = a_0 + a_1x + \dots + a_nx^n$

## Subspaces of vector spaces

*Definition.* A vector space  $V_0$  is a **subspace** of a vector space  $V$  if  $V_0 \subset V$  and the linear operations on  $V_0$  agree with the linear operations on  $V$ .

**Proposition** A subset  $S$  of a vector space  $V$  is a subspace of  $V$  if and only if  $S$  is **nonempty** and **closed under linear operations**, i.e.,

$$\mathbf{x}, \mathbf{y} \in S \implies \mathbf{x} + \mathbf{y} \in S,$$

$$\mathbf{x} \in S \implies r\mathbf{x} \in S \text{ for all } r \in \mathbb{R}.$$

*Remarks.* The zero vector in a subspace is the same as the zero vector in  $V$ . Also, the subtraction in a subspace agrees with that in  $V$ .

## Examples of subspaces

- $F(\mathbb{R})$ : all functions  $f : \mathbb{R} \rightarrow \mathbb{R}$
- $C(\mathbb{R})$ : all continuous functions  $f : \mathbb{R} \rightarrow \mathbb{R}$

$C(\mathbb{R})$  is a subspace of  $F(\mathbb{R})$ .

- $\mathcal{P}$ : polynomials  $p(x) = a_0 + a_1x + \cdots + a_{n-1}x^{n-1}$
- $\mathcal{P}_n$ : polynomials of degree **less than**  $n$

$\mathcal{P}_n$  is a subspace of  $\mathcal{P}$ .

- Any vector space  $V$
- $\{\mathbf{0}\}$ , where  $\mathbf{0}$  is the zero vector in  $V$

The trivial space  $\{\mathbf{0}\}$  is a subspace of  $V$ .

*Example.*  $V = \mathbb{R}^2$ .

- The line  $x - y = 0$  is a subspace of  $\mathbb{R}^2$ .

The line consists of all vectors of the form  $(t, t)$ ,  $t \in \mathbb{R}$ .

$$(t, t) + (s, s) = (t + s, t + s) \implies \text{closed under addition}$$
$$r(t, t) = (rt, rt) \implies \text{closed under scaling}$$

- The parabola  $y = x^2$  is not a subspace of  $\mathbb{R}^2$ .

It is enough to find one explicit counterexample.

*Counterexample 1:*  $(1, 1) + (-1, 1) = (0, 2)$ .

$(1, 1)$  and  $(-1, 1)$  lie on the parabola while  $(0, 2)$  does not  
 $\implies$  not closed under addition

*Counterexample 2:*  $2(1, 1) = (2, 2)$ .

$(1, 1)$  lies on the parabola while  $(2, 2)$  does not  
 $\implies$  not closed under scaling

*Example.*  $V = \mathbb{R}^3$ .

- The plane  $z = 0$  is a subspace of  $\mathbb{R}^3$ .
- The plane  $z = 1$  is not a subspace of  $\mathbb{R}^3$ .
- The line  $t(1, 1, 0)$ ,  $t \in \mathbb{R}$  is a subspace of  $\mathbb{R}^3$  and a subspace of the plane  $z = 0$ .
- The line  $(1, 1, 1) + t(1, -1, 0)$ ,  $t \in \mathbb{R}$  is not a subspace of  $\mathbb{R}^3$  as it lies in the plane  $x + y + z = 3$ , which does not contain  $\mathbf{0}$ .
- In general, a line or a plane in  $\mathbb{R}^3$  is a subspace if and only if it passes through the origin.



System of linear equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

Any solution  $(x_1, x_2, \dots, x_n)$  is an element of  $\mathbb{R}^n$ .

**Theorem** The solution set of the system is a subspace of  $\mathbb{R}^n$  if and only if all  $b_i = 0$ .

**Theorem** The solution set of a system of linear equations in  $n$  variables is a subspace of  $\mathbb{R}^n$  if and only if all equations are homogeneous.

*Proof:* “only if”: the zero vector  $\mathbf{0} = (0, 0, \dots, 0)$  is a solution only if all equations are homogeneous.

“if”: a system of homogeneous linear equations is equivalent to a matrix equation  $A\mathbf{x} = \mathbf{0}$ .

$A\mathbf{0} = \mathbf{0} \implies \mathbf{0}$  is a solution  $\implies$  solution set is not empty.

If  $A\mathbf{x} = \mathbf{0}$  and  $A\mathbf{y} = \mathbf{0}$  then  $A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \mathbf{0}$ .

If  $A\mathbf{x} = \mathbf{0}$  then  $A(r\mathbf{x}) = r(A\mathbf{x}) = \mathbf{0}$ .

Examples of subspaces of  $\mathcal{M}_{2,2}(\mathbb{R})$ :  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

- diagonal matrices:  $b = c = 0$
- upper triangular matrices:  $c = 0$
- lower triangular matrices:  $b = 0$
- symmetric matrices ( $A^T = A$ ):  $b = c$
- anti-symmetric matrices ( $A^T = -A$ ):

$$a = d = 0, \quad c = -b$$

- matrices with zero trace:  $a + d = 0$   
(trace = the sum of diagonal entries)

- matrices with zero determinant,  $ad - bc = 0$ ,

**do not** form a subspace:  $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ .

Let  $V$  be a vector space and  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n \in V$ . Consider the set  $L$  of all linear combinations  $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \dots + r_n\mathbf{v}_n$ , where  $r_1, r_2, \dots, r_n \in \mathbb{R}$ .

**Theorem**  $L$  is a subspace of  $V$ .

*Proof:* First of all,  $L$  is not empty. For example,  $\mathbf{0} = 0\mathbf{v}_1 + 0\mathbf{v}_2 + \dots + 0\mathbf{v}_n$  belongs to  $L$ .

The set  $L$  is closed under addition since

$$\begin{aligned}(r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \dots + r_n\mathbf{v}_n) + (s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + \dots + s_n\mathbf{v}_n) &= \\ &= (r_1 + s_1)\mathbf{v}_1 + (r_2 + s_2)\mathbf{v}_2 + \dots + (r_n + s_n)\mathbf{v}_n.\end{aligned}$$

The set  $L$  is closed under scalar multiplication since

$$t(r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \dots + r_n\mathbf{v}_n) = (tr_1)\mathbf{v}_1 + (tr_2)\mathbf{v}_2 + \dots + (tr_n)\mathbf{v}_n.$$

## Span: implicit definition

Let  $S$  be a subset of a vector space  $V$ .

*Definition.* The **span** of the set  $S$ , denoted  $\text{Span}(S)$ , is the smallest subspace of  $V$  that contains  $S$ . That is,

- $\text{Span}(S)$  is a subspace of  $V$ ;
- for any subspace  $W \subset V$  one has
$$S \subset W \implies \text{Span}(S) \subset W.$$

*Remark.* The span of any set  $S \subset V$  is well defined (it is the intersection of all subspaces of  $V$  that contain  $S$ ).

## Span: effective description

Let  $S$  be a subset of a vector space  $V$ .

- If  $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  then  $\text{Span}(S)$  is the set of all linear combinations  $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \dots + r_n\mathbf{v}_n$ , where  $r_1, r_2, \dots, r_n \in \mathbb{R}$ .
- If  $S$  is an infinite set then  $\text{Span}(S)$  is the set of all linear combinations  $r_1\mathbf{u}_1 + r_2\mathbf{u}_2 + \dots + r_k\mathbf{u}_k$ , where  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k \in S$  and  $r_1, r_2, \dots, r_k \in \mathbb{R}$  ( $k \geq 1$ ).
- If  $S$  is the empty set then  $\text{Span}(S) = \{\mathbf{0}\}$ .

*Examples of subspaces of  $\mathcal{M}_{2,2}(\mathbb{R})$ :*

- The span of  $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$  and  $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$  consists of all matrices of the form

$$a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

This is the subspace of diagonal matrices.

- The span of  $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ ,  $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ , and  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  consists of all matrices of the form

$$a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + c \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a & c \\ c & b \end{pmatrix}.$$

This is the subspace of symmetric matrices.

*Examples of subspaces of  $\mathcal{M}_{2,2}(\mathbb{R})$ :*

- The span of  $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  is the subspace of anti-symmetric matrices.
- The span of  $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ ,  $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ , and  $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$  is the subspace of upper triangular matrices.
- The span of  $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ ,  $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ ,  $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ ,  $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$  is the entire space  $\mathcal{M}_{2,2}(\mathbb{R})$ .



## Spanning set

*Definition.* A subset  $S$  of a vector space  $V$  is called a **spanning set** for  $V$  if  $\text{Span}(S) = V$ .

*Examples.*

- Vectors  $\mathbf{e}_1 = (1, 0, 0)$ ,  $\mathbf{e}_2 = (0, 1, 0)$ , and  $\mathbf{e}_3 = (0, 0, 1)$  form a spanning set for  $\mathbb{R}^3$  as

$$(x, y, z) = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3.$$

- Matrices  $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ ,  $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ ,  $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ ,  $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

form a spanning set for  $\mathcal{M}_{2,2}(\mathbb{R})$  as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

**Problem** Let  $\mathbf{v}_1 = (1, 2, 0)$ ,  $\mathbf{v}_2 = (3, 1, 1)$ , and  $\mathbf{w} = (4, -7, 3)$ . Determine whether  $\mathbf{w}$  belongs to  $\text{Span}(\mathbf{v}_1, \mathbf{v}_2)$ .

We have to check if there exist  $r_1, r_2 \in \mathbb{R}$  such that  $\mathbf{w} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2$ . This vector equation is equivalent to a system of linear equations:

$$\begin{cases} 4 = r_1 + 3r_2 \\ -7 = 2r_1 + r_2 \\ 3 = 0r_1 + r_2 \end{cases} \iff \begin{cases} r_1 = -5 \\ r_2 = 3 \end{cases}$$

Thus  $\mathbf{w} = -5\mathbf{v}_1 + 3\mathbf{v}_2 \in \text{Span}(\mathbf{v}_1, \mathbf{v}_2)$ .

**Problem** Let  $\mathbf{v}_1 = (2, 5)$  and  $\mathbf{v}_2 = (1, 3)$ . Show that  $\{\mathbf{v}_1, \mathbf{v}_2\}$  is a spanning set for  $\mathbb{R}^2$ .

Take any vector  $\mathbf{w} = (a, b) \in \mathbb{R}^2$ . We have to check that there exist  $r_1, r_2 \in \mathbb{R}$  such that

$$\mathbf{w} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 \iff \begin{cases} 2r_1 + r_2 = a \\ 5r_1 + 3r_2 = b \end{cases}$$

Coefficient matrix:  $C = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$ .  $\det C = 1 \neq 0$ .

Since the matrix  $C$  is invertible, the system has a unique solution for any  $a$  and  $b$ .

Thus  $\text{Span}(\mathbf{v}_1, \mathbf{v}_2) = \mathbb{R}^2$ .

**Problem** Let  $\mathbf{v}_1 = (2, 5)$  and  $\mathbf{v}_2 = (1, 3)$ . Show that  $\{\mathbf{v}_1, \mathbf{v}_2\}$  is a spanning set for  $\mathbb{R}^2$ .

*Alternative solution:* First let us show that vectors  $\mathbf{e}_1 = (1, 0)$  and  $\mathbf{e}_2 = (0, 1)$  belong to  $\text{Span}(\mathbf{v}_1, \mathbf{v}_2)$ .

$$\mathbf{e}_1 = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 \iff \begin{cases} 2r_1 + r_2 = 1 \\ 5r_1 + 3r_2 = 0 \end{cases} \iff \begin{cases} r_1 = 3 \\ r_2 = -5 \end{cases}$$

$$\mathbf{e}_2 = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 \iff \begin{cases} 2r_1 + r_2 = 0 \\ 5r_1 + 3r_2 = 1 \end{cases} \iff \begin{cases} r_1 = -1 \\ r_2 = 2 \end{cases}$$

Thus  $\mathbf{e}_1 = 3\mathbf{v}_1 - 5\mathbf{v}_2$  and  $\mathbf{e}_2 = -\mathbf{v}_1 + 2\mathbf{v}_2$ .

Then for any vector  $\mathbf{w} = (a, b) \in \mathbb{R}^2$  we have

$$\begin{aligned} \mathbf{w} &= a\mathbf{e}_1 + b\mathbf{e}_2 = a(3\mathbf{v}_1 - 5\mathbf{v}_2) + b(-\mathbf{v}_1 + 2\mathbf{v}_2) \\ &= (3a - b)\mathbf{v}_1 + (-5a + 2b)\mathbf{v}_2. \end{aligned}$$

**Problem** Let  $\mathbf{v}_1 = (2, 5)$  and  $\mathbf{v}_2 = (1, 3)$ . Show that  $\{\mathbf{v}_1, \mathbf{v}_2\}$  is a spanning set for  $\mathbb{R}^2$ .

*Remarks on the alternative solution:*

Notice that  $\mathbb{R}^2$  is spanned by vectors  $\mathbf{e}_1 = (1, 0)$  and  $\mathbf{e}_2 = (0, 1)$  since  $(a, b) = a\mathbf{e}_1 + b\mathbf{e}_2$ .

This is why we have checked that vectors  $\mathbf{e}_1$  and  $\mathbf{e}_2$  belong to  $\text{Span}(\mathbf{v}_1, \mathbf{v}_2)$ . Then

$$\begin{aligned} \mathbf{e}_1, \mathbf{e}_2 \in \text{Span}(\mathbf{v}_1, \mathbf{v}_2) &\implies \text{Span}(\mathbf{e}_1, \mathbf{e}_2) \subset \text{Span}(\mathbf{v}_1, \mathbf{v}_2) \\ &\implies \mathbb{R}^2 \subset \text{Span}(\mathbf{v}_1, \mathbf{v}_2) \implies \text{Span}(\mathbf{v}_1, \mathbf{v}_2) = \mathbb{R}^2. \end{aligned}$$

In general, to show that  $\text{Span}(S_1) = \text{Span}(S_2)$ , it is enough to check that  $S_1 \subset \text{Span}(S_2)$  and  $S_2 \subset \text{Span}(S_1)$ .