

MATH 304
Linear Algebra

Lecture 23:
Matrix exponentials.

Initial value problem for a system of linear PDEs:

$$\begin{cases} \frac{dx}{dt} = 4x + 3y, \\ \frac{dy}{dt} = y, \end{cases} \quad x(0) = 1, \quad y(0) = 1.$$

The system can be rewritten in vector form:

$$\frac{d\mathbf{v}}{dt} = A\mathbf{v}, \quad \text{where } A = \begin{pmatrix} 4 & 3 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Matrix A is diagonalizable: $A = UBU^{-1}$, where

$$B = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Let $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ be coordinates of the vector \mathbf{v} relative to the basis $\mathbf{v}_1 = (1, 0)$, $\mathbf{v}_2 = (-1, 1)$ of eigenvectors of A . Then $\mathbf{v} = U\mathbf{w} \implies \mathbf{w} = U^{-1}\mathbf{v}$.

It follows that

$$\frac{d\mathbf{w}}{dt} = \frac{d}{dt}(U^{-1}\mathbf{v}) = U^{-1}\frac{d\mathbf{v}}{dt} = U^{-1}A\mathbf{v} = U^{-1}AU\mathbf{w}.$$

$$\text{Hence } \frac{d\mathbf{w}}{dt} = B\mathbf{w} \iff \begin{cases} \frac{dw_1}{dt} = 4w_1, \\ \frac{dw_2}{dt} = w_2. \end{cases}$$

General solution: $w_1(t) = c_1 e^{4t}$, $w_2(t) = c_2 e^t$, where $c_1, c_2 \in \mathbb{R}$.

Initial condition:

$$\mathbf{w}(0) = U^{-1}\mathbf{v}(0) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Thus $w_1(t) = 2e^{4t}$, $w_2(t) = e^t$. Then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = U\mathbf{w}(t) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2e^{4t} \\ e^t \end{pmatrix} = \begin{pmatrix} 2e^{4t} - e^t \\ e^t \end{pmatrix}.$$

- *Initial value problem for a linear ODE:*

$$\frac{dy}{dt} = 2y, \quad y(0) = 3.$$

Solution: $y(t) = 3e^{2t}$.

- *Initial value problem for a system of linear ODEs:*

$$\begin{cases} \frac{dx}{dt} = 2x + 3y, \\ \frac{dy}{dt} = x + 4y, \end{cases} \quad x(0) = 2, \quad y(0) = 1.$$

The system can be rewritten in vector form

$$\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{where } A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}.$$

Solution: $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{tA} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

What is e^{tA} ?

Exponential function

Exponential function: $f(x) = \exp x = e^x$, $x \in \mathbb{R}$.

Principal property: $e^{x+y} = e^x \cdot e^y$.

Definition 1. $e^x = \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n$.

In particular, $e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n \approx 2.7182818$.

Definition 2. $e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$

Definition 3. $f(x) = e^x$ is the unique solution of the initial value problem $f' = f$, $f(0) = 1$.

Complex exponentials

Definition. For any $z \in \mathbb{C}$ let

$$e^z = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots$$

Remark. A sequence of complex numbers $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2, \dots$ converges to $z = x + iy$ if $x_n \rightarrow x$ and $y_n \rightarrow y$ as $n \rightarrow \infty$.

Theorem 1 If $z = x + iy, x, y \in \mathbb{R}$, then

$$e^z = e^x(\cos y + i \sin y).$$

In particular, $e^{i\phi} = \cos \phi + i \sin \phi, \phi \in \mathbb{R}$.

Theorem 2 $e^{z+w} = e^z \cdot e^w$ for all $z, w \in \mathbb{C}$.

Proposition $e^{i\phi} = \cos \phi + i \sin \phi$ for all $\phi \in \mathbb{R}$.

Proof:
$$e^{i\phi} = 1 + i\phi + \frac{(i\phi)^2}{2!} + \dots + \frac{(i\phi)^n}{n!} + \dots$$

The sequence $1, i, i^2, i^3, \dots, i^n, \dots$ is periodic:

$$\underbrace{1, i, -1, -i}, \underbrace{1, i, -1, -i}, \dots$$

It follows that

$$\begin{aligned} e^{i\phi} &= 1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} - \dots + (-1)^k \frac{\phi^{2k}}{(2k)!} + \dots \\ &+ i \left(\phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - \dots + (-1)^k \frac{\phi^{2k+1}}{(2k+1)!} + \dots \right) \\ &= \cos \phi + i \sin \phi. \end{aligned}$$

Matrix exponentials

Definition. For any square matrix A let

$$\exp A = e^A = I + A + \frac{1}{2!}A^2 + \cdots + \frac{1}{n!}A^n + \cdots$$

Matrix exponential is a limit of matrix polynomials.

Remark. Let $A^{(1)}, A^{(2)}, \dots$ be a sequence of $n \times n$ matrices, $A^{(n)} = (a_{ij}^{(n)})$. The sequence converges to an $n \times n$ matrix $B = (b_{ij})$ if $a_{ij}^{(n)} \rightarrow b_{ij}$ as $n \rightarrow \infty$, i.e., if each entry converges.

Theorem The matrix $\exp A$ is well defined, i.e., the series converges.

Properties of matrix exponentials

Theorem 1 If $AB = BA$ then $e^A e^B = e^B e^A = e^{A+B}$.

Corollary (a) $e^{tA} e^{sA} = e^{sA} e^{tA} = e^{(t+s)A}$, $t, s \in \mathbb{R}$;

(b) $e^O = I$; **(c)** $(e^A)^{-1} = e^{-A}$.

Theorem 2 $\frac{d}{dt} e^{tA} = A e^{tA} = e^{tA} A$.

Indeed, $e^{tA} = I + tA + \frac{t^2}{2!} A^2 + \cdots + \frac{t^n}{n!} A^n + \cdots$,

and the series can be differentiated term by term.

$$\frac{d}{dt} \left(\frac{t^n}{n!} A^n \right) = \frac{d}{dt} \left(\frac{t^n}{n!} \right) A^n = \frac{t^{n-1}}{(n-1)!} A^n.$$

Lemma Let A be an $n \times n$ matrix and $\mathbf{x} \in \mathbb{R}^n$. Then the vector function $\mathbf{v}(t) = e^{tA}\mathbf{x}$ satisfies $\mathbf{v}' = A\mathbf{v}$.

Proof:
$$\frac{d\mathbf{v}}{dt} = \frac{d}{dt}e^{tA}\mathbf{x} = (Ae^{tA})\mathbf{x} = A(e^{tA}\mathbf{x}) = A\mathbf{v}.$$

Theorem For any $t_0 \in \mathbb{R}$ and $\mathbf{x}_0 \in \mathbb{R}^n$ the initial value problem

$$\frac{d\mathbf{v}}{dt} = A\mathbf{v}, \quad \mathbf{v}(t_0) = \mathbf{x}_0$$

has a unique solution $\mathbf{v}(t) = e^{(t-t_0)A}\mathbf{x}_0$.

Indeed, $\mathbf{v}(t) = e^{(t-t_0)A}\mathbf{x}_0 = e^{tA}e^{-t_0A}\mathbf{x}_0 = e^{tA}\mathbf{x}$, where $\mathbf{x} = e^{-t_0A}\mathbf{x}_0$ is a constant vector.

Evaluation of matrix exponentials

Example. $A = \text{diag}(a_1, a_2, \dots, a_k)$.

$$A^n = \text{diag}(a_1^n, a_2^n, \dots, a_k^n), \quad n = 1, 2, 3, \dots$$

$$\begin{aligned} e^A &= I + A + \frac{1}{2!}A^2 + \dots + \frac{1}{n!}A^n + \dots \\ &= \text{diag}(b_1, b_2, \dots, b_k), \end{aligned}$$

where $b_i = 1 + a_i + \frac{1}{2!}a_i^2 + \frac{1}{3!}a_i^3 + \dots = e^{a_i}$.

Theorem If $A = \text{diag}(a_1, a_2, \dots, a_k)$ then

$$\begin{aligned} e^A &= \text{diag}(e^{a_1}, e^{a_2}, \dots, e^{a_k}), \\ e^{tA} &= \text{diag}(e^{a_1 t}, e^{a_2 t}, \dots, e^{a_k t}). \end{aligned}$$

Let A be an n -by- n matrix and suppose there exists a basis $\mathbf{v}_1, \dots, \mathbf{v}_n$ for \mathbb{R}^n consisting of eigenvectors of A . That is, $A\mathbf{v}_k = \lambda_k\mathbf{v}_k$, where $\lambda_k \in \mathbb{R}$.

Then $A = UBU^{-1}$, where $B = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ and U is a transition matrix whose columns are vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$.

$$A^2 = UBU^{-1}UBU^{-1} = UB^2U^{-1},$$

$$A^3 = A^2A = UB^2U^{-1}UBU^{-1} = UB^3U^{-1}.$$

Likewise, $A^n = UB^nU^{-1}$ for any $n \geq 1$.

$$\begin{aligned} I + 2A - 3A^2 &= UIU^{-1} + 2UBU^{-1} - 3UB^2U^{-1} = \\ &= U(I + 2B - 3B^2)U^{-1}. \end{aligned}$$

$$\implies p(A) = Up(B)U^{-1} \text{ for any polynomial } p(x).$$

$$\implies e^{tA} = Ue^{tB}U^{-1} \text{ for all } t \in \mathbb{R}.$$

Example. $A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$.

The eigenvalues of A : $\lambda_1 = 1, \lambda_2 = 5$.

Eigenvectors: $\mathbf{v}_1 = (3, -1), \mathbf{v}_2 = (1, 1)$.

Therefore $A = UBU^{-1}$, where

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}, \quad U = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}.$$

Then

$$\begin{aligned} e^{tA} &= Ue^{tB}U^{-1} = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & e^{5t} \end{pmatrix} \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 3e^t & e^{5t} \\ -e^t & e^{5t} \end{pmatrix} \frac{1}{4} \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 3e^t + e^{5t} & -3e^t + 3e^{5t} \\ -e^t + e^{5t} & e^t + 3e^{5t} \end{pmatrix}. \end{aligned}$$

Problem. Solve a system of differential equations

$$\begin{cases} \frac{dx}{dt} = 2x + 3y, \\ \frac{dy}{dt} = x + 4y \end{cases}$$

subject to initial conditions $x(0) = 2$, $y(0) = 1$.

The unique solution:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{tA} \mathbf{x}_0, \text{ where } A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}, \mathbf{x}_0 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

$$e^{tA} = \frac{1}{4} \begin{pmatrix} 3e^t + e^{5t} & -3e^t + 3e^{5t} \\ -e^t + e^{5t} & e^t + 3e^{5t} \end{pmatrix}$$

$$\implies \begin{cases} x(t) = \frac{3}{4}e^t + \frac{5}{4}e^{5t}, \\ y(t) = -\frac{1}{4}e^t + \frac{5}{4}e^{5t}. \end{cases}$$

Example. $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, a Jordan block.

$$A^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A^3 = O, \quad A^n = O \text{ for } n \geq 3.$$

$$e^A = I + A + \frac{1}{2}A^2 = \begin{pmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$e^{tA} = I + tA + \frac{t^2}{2}A^2 = \begin{pmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}.$$

Example. $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, another Jordan block.

$$A^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \quad A^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$

$$e^{tA} = I + tA + \frac{t^2}{2!}A^2 + \cdots + \frac{t^n}{n!}A^n + \cdots = \begin{pmatrix} a(t) & b(t) \\ 0 & a(t) \end{pmatrix},$$

where $a(t) = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots = e^t$,

$$b(t) = t + 2\frac{t^2}{2!} + 3\frac{t^3}{3!} + \cdots = te^t.$$

$$\text{Thus } e^{tA} = \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix}.$$

Example. $A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, a general Jordan block.

We have that $A = \lambda I + B$, where $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Since $(\lambda I)B = B(\lambda I)$, it follows that $e^A = e^{\lambda I} e^B$.
Similarly, $e^{tA} = e^{t\lambda I} e^{tB}$.

$$e^{t\lambda I} = \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{\lambda t} \end{pmatrix} = e^{\lambda t} I,$$

$$B^2 = O \implies e^{tB} = I + tB = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

$$\text{Thus } e^{tA} = e^{t\lambda I} e^{tB} = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}.$$

Problem. Solve a system of differential equations

$$\begin{cases} \frac{dx}{dt} = 2x + y, \\ \frac{dy}{dt} = 2y \end{cases}$$

subject to initial conditions $x(0) = y(0) = 1$.

The unique solution:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{tA} \mathbf{x}_0, \text{ where } A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \mathbf{x}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$$e^{tA} = \begin{pmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{pmatrix} = e^{2t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

$$\implies \begin{cases} x(t) = e^{2t}(1 + t), \\ y(t) = e^{2t}. \end{cases}$$