MATH 304

Linear Algebra

Lecture 26:
Review for the final exam.

Topics for the final exam: Part I

Elementary linear algebra (Leon 1.1–1.4, 2.1–2.2)

- Systems of linear equations: elementary operations, Gaussian elimination, back substitution.

- Matrix of coefficients and augmented matrix. Elementary row operations, row echelon form and reduced row echelon form.

- Matrix algebra. Inverse matrix.

- Determinants: explicit formulas for 2×2 and 3×3 matrices, row and column expansions, elementary row and column operations.

Topics for the final exam: Part II

Abstract linear algebra (Leon 3.1–3.6, 4.1–4.3)

- Vector spaces (vectors, matrices, polynomials, functional spaces).
- Subspaces. Nullspace, column space, and row space of a matrix.
- Span, spanning set. Linear independence.
- Bases and dimension.
- Rank and nullity of a matrix.
- Coordinates relative to a basis.
- Change of basis, transition matrix.

- Linear transformations.
- Matrix transformations.
- Matrix of a linear mapping.
- Similarity of matrices.

Topics for the final exam: Parts III–IV

Advanced linear algebra (Leon 5.1–5.7, 6.1–6.3)

- Euclidean structure in $\mathbb{R}^n$ (length, angle, dot product)
- Inner products and norms
- Orthogonal complement
- Least squares problems
- The Gram-Schmidt orthogonalization process
- Orthogonal polynomials

- Eigenvalues, eigenvectors, eigenspaces
- Characteristic polynomial
- Bases of eigenvectors, diagonalization
- Orthogonal matrices
- Rotations in space
- Matrix exponentials

Bases of eigenvectors

Let A be an $n \times n$ matrix with real entries.

- A has n distinct real eigenvalues $\implies$ a basis for $\mathbb{R}^n$ formed by eigenvectors of A
- A has complex eigenvalues $\implies$ no basis for $\mathbb{R}^n$ formed by eigenvectors of A
- A has n distinct complex eigenvalues $\implies$ a basis for $\mathbb{C}^n$ formed by eigenvectors of A
- A has multiple eigenvalues $\implies$ further information is needed
- an orthonormal basis for $\mathbb{R}^n$ formed by eigenvectors of A $\iff$ A is symmetric: $A^T = A$

Problem. For each of the following matrices determine whether it allows

(a) a basis of eigenvectors for $\mathbb{R}^n$,
(b) a basis of eigenvectors for $\mathbb{C}^n$,
(c) an orthonormal basis of eigenvectors for $\mathbb{R}^n$.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$$ (a),(b),(c): yes

$$B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$ (a),(b),(c): no

Problem. For each of the following matrices determine whether it allows

(a) a basis of eigenvectors for $\mathbb{R}^n$,
(b) a basis of eigenvectors for $\mathbb{C}^n$,
(c) an orthonormal basis of eigenvectors for $\mathbb{R}^n$.

$C = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$ (a),(b): yes (c): no

$D = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (b): yes (a),(c): no

Problem Let V be the vector space spanned by functions $f_1(x) = x \sin x$, $f_2(x) = x \cos x$, $f_3(x) = \sin x$, and $f_4(x) = \cos x$.
Consider the linear operator $D : V \to V$, $D = d/dx$.

(a) Find the matrix A of the operator D relative to the basis f_1, f_2, f_3, f_4.
(b) Find the eigenvalues of A.
(c) Is the matrix A diagonalizable in $\mathbb{R}^4$ (in $\mathbb{C}^4$)?

A is a 4×4 matrix whose columns are coordinates of functions $Df_i = f_i'$ relative to the basis f_1, f_2, f_3, f_4.

$$f_1'(x) = (x \sin x)' = x \cos x + \sin x = f_2(x) + f_3(x),$$
$$f_2'(x) = (x \cos x)' = -x \sin x + \cos x$$
$$= -f_1(x) + f_4(x),$$
$$f_3'(x) = (\sin x)' = \cos x = f_4(x),$$
$$f_4'(x) = (\cos x)' = -\sin x = -f_3(x).$$

Thus $A = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$

Eigenvalues of A are roots of its characteristic polynomial

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & -1 & 0 & 0 \\ 1 & -\lambda & 0 & 0 \\ 1 & 0 & -\lambda & -1 \\ 0 & 1 & 1 & -\lambda \end{vmatrix}$$

Expand the determinant by the 1st row:

$$\det(A - \lambda I) = -\lambda \begin{vmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & -1 \\ 1 & 1 & -\lambda \end{vmatrix} - (-1) \begin{vmatrix} 1 & 0 & 0 \\ 1 & -\lambda & -1 \\ 0 & 1 & -\lambda \end{vmatrix}$$

$$= \lambda^2(\lambda^2 + 1) + (\lambda^2 + 1) = (\lambda^2 + 1)^2.$$

The eigenvalues are i and $-i$, both of multiplicity 2.

Complex eigenvalues $\implies$ A is not diagonalizable in $\mathbb{R}^4$

If A is diagonalizable in $\mathbb{C}^4$ then $A = UXU^{-1}$, where U is an invertible matrix with complex entries and

$$X = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}.$$

This would imply that $A^2 = UX^2U^{-1}$. But $X^2 = -I$ so that $A^2 = U(-I)U^{-1} = -I$.

$$A^2 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -2 & -1 & 0 \\ 2 & 0 & 0 & -1 \end{pmatrix}.$$

Since $A^2 \neq -I$, the matrix A is not diagonalizable in $\mathbb{C}^4$.

Problem Consider a linear operator $L : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $L(\mathbf{v}) = \mathbf{v}_0 \times \mathbf{v}$, where $\mathbf{v}_0 = (3/5, 0, -4/5)$.

(a) Find the matrix B of the operator L.
(b) Find the range and kernel of L.
(c) Find the eigenvalues of L.
(d) Find the matrix of the operator L^{2010} (L applied 2010 times).

$$L(\mathbf{v}) = \mathbf{v}_0 \times \mathbf{v}, \quad \mathbf{v}_0 = (3/5, 0, -4/5).$$

Let $\mathbf{v} = (x, y, z) = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$. Then

$$L(\mathbf{v}) = \mathbf{v}_0 \times \mathbf{v} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 3/5 & 0 & -4/5 \\ x & y & z \end{vmatrix}$$

$$= \tfrac{4}{5}y\mathbf{e}_1 - \left(\tfrac{4}{5}x + \tfrac{3}{5}z\right)\mathbf{e}_2 + \tfrac{3}{5}y\mathbf{e}_3.$$

In particular, $L(\mathbf{e}_1) = -\frac{4}{5}\mathbf{e}_2$, $L(\mathbf{e}_2) = \frac{4}{5}\mathbf{e}_1 + \frac{3}{5}\mathbf{e}_3$, $L(\mathbf{e}_3) = -\frac{3}{5}\mathbf{e}_2$.

Therefore $B = \begin{pmatrix} 0 & 4/5 & 0 \\ -4/5 & 0 & -3/5 \\ 0 & 3/5 & 0 \end{pmatrix}.$

$$B = \begin{pmatrix} 0 & 4/5 & 0 \\ -4/5 & 0 & -3/5 \\ 0 & 3/5 & 0 \end{pmatrix}.$$

The range of the operator L is spanned by columns of the matrix B. It follows that $\mathrm{Range}(L)$ is the plane spanned by $\mathbf{v}_1 = (0,1,0)$ and $\mathbf{v}_2 = (4,0,3)$.

The kernel of L is the nullspace of the matrix B, i.e., the solution set for the equation $B\mathbf{x} = \mathbf{0}$.

$$\begin{pmatrix} 0 & 4/5 & 0 \\ -4/5 & 0 & -3/5 \\ 0 & 3/5 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 3/4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Longrightarrow \ x + \tfrac{3}{4}z = y = 0 \ \Longrightarrow \ \mathbf{x} = t(-3/4, 0, 1).$$

Alternatively, the kernel of L is the set of vectors $\mathbf{v} \in \mathbb{R}^3$ such that $L(\mathbf{v}) = \mathbf{v}_0 \times \mathbf{v} = \mathbf{0}$.
It follows that this is the line spanned by $\mathbf{v}_0 = (3/5, 0, -4/5)$.

Characteristic polynomial of the matrix B:

$$\det(B - \lambda I) = \begin{vmatrix} -\lambda & 4/5 & 0 \\ -4/5 & -\lambda & -3/5 \\ 0 & 3/5 & -\lambda \end{vmatrix}$$

$$= -\lambda^3 - (3/5)^2\lambda - (4/5)^2\lambda = -\lambda^3 - \lambda = -\lambda(\lambda^2 + 1).$$

The eigenvalues are 0, i, and $-i$.

The matrix of the operator L^{2010} is B^{2010}.

Since the matrix B has eigenvalues 0, i, and $-i$, it is diagonalizable in $\mathbb{C}^3$. Namely, $B = UDU^{-1}$, where U is an invertible matrix with complex entries and

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}.$$

Then $B^{2010} = UD^{2010}U^{-1}$. We have that $D^{2010} =$
$= \mathrm{diag}\big(0, i^{2010}, (-i)^{2010}\big) = \mathrm{diag}(0, -1, -1) = D^2$.
Hence

$$B^{2010} = UD^2U^{-1} = B^2 = \begin{pmatrix} -0.64 & 0 & -0.48 \\ 0 & -1 & 0 \\ -0.48 & 0 & -0.36 \end{pmatrix}.$$

Problem. Let $f_1, f_2, f_3, \ldots$ be the Fibonacci numbers defined by $f_1 = f_2 = 1$, $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$. Find $\lim\limits_{n\to\infty} \dfrac{f_{n+1}}{f_n}$.

For any integer $n \geq 1$,

$$\begin{pmatrix} f_{n+2} \\ f_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix}.$$

That is, $\mathbf{v}_{n+1} = A\mathbf{v}_n$, where $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $\mathbf{v}_n = \begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix}$.

In particular, $\mathbf{v}_2 = A\mathbf{v}_1$, $\mathbf{v}_3 = A\mathbf{v}_2 = A^2\mathbf{v}_1$, $\mathbf{v}_4 = A\mathbf{v}_3 = A^3\mathbf{v}_1$. In general, $\mathbf{v}_n = A^{n-1}\mathbf{v}_1$.

Characteristic equation of the matrix A:

$$\begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0 \iff \lambda^2 - \lambda - 1 = 0.$$

Eigenvalues: $\lambda_1 = \frac{1+\sqrt{5}}{2}$, $\lambda_2 = \frac{1-\sqrt{5}}{2}$.

Let $\mathbf{w}_1 = (x_1, y_1)$ and $\mathbf{w}_2 = (x_2, y_2)$ be eigenvectors of A associated with the eigenvalues λ_1 and λ_2. Then $\mathbf{w}_1, \mathbf{w}_2$ is a basis for $\mathbb{R}^2$.

In particular, $\mathbf{v}_1 = (1,1) = c_1\mathbf{w}_1 + c_2\mathbf{w}_2$ for some $c_1, c_2 \in \mathbb{R}$. It follows that

$$\mathbf{v}_n = A^{n-1}\mathbf{v}_1 = A^{n-1}(c_1\mathbf{w}_1 + c_2\mathbf{w}_2)$$

$$= c_1 A^{n-1}\mathbf{w}_1 + c_2 A^{n-1}\mathbf{w}_2 = c_1\lambda_1^{n-1}\mathbf{w}_1 + c_2\lambda_2^{n-1}\mathbf{w}_2.$$

$$\mathbf{v}_n = c_1\lambda_1^{n-1}\mathbf{w}_1 + c_2\lambda_2^{n-1}\mathbf{w}_2$$

$$\implies \quad f_n = c_1\lambda_1^{n-1}y_1 + c_2\lambda_2^{n-1}y_2.$$

Recall that $\lambda_1 = \frac{1+\sqrt{5}}{2}$, $\lambda_2 = \frac{1-\sqrt{5}}{2}$.

We have $\lambda_1 > 1$ and $-1 < \lambda_2 < 0$.

Therefore

$$\frac{f_{n+1}}{f_n} = \frac{c_1\lambda_1^n y_1 + c_2\lambda_2^n y_2}{c_1\lambda_1^{n-1}y_1 + c_2\lambda_2^{n-1}y_2}$$

$$= \lambda_1 \frac{c_1y_1 + c_2(\lambda_2/\lambda_1)^n y_2}{c_1y_1 + c_2(\lambda_2/\lambda_1)^{n-1}y_2} \to \lambda_1 \frac{c_1y_1}{c_1y_1} = \lambda_1$$

provided that $c_1y_1 \neq 0$.

Thus $\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \lambda_1 = \frac{1+\sqrt{5}}{2}$.