MATH 304

Linear Algebra

Lecture 5:
Matrix algebra.

Matrices

Definition. An **m-by-n matrix** is a rectangular array of numbers that has m rows and n columns:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

Notation: $A = (a_{ij})_{1 \le i \le n,\, 1 \le j \le m}$ or simply $A = (a_{ij})$ if the dimensions are known.

An n-dimensional vector can be represented as a $1 \times n$ matrix (row vector) or as an $n \times 1$ matrix (column vector):

$$(x_1, x_2, \ldots, x_n) \qquad \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

An $m \times n$ matrix $A = (a_{ij})$ can be regarded as a column of n-dimensional row vectors or as a row of m-dimensional column vectors:

$$A = \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_m \end{pmatrix}, \qquad \mathbf{v}_i = (a_{i1}, a_{i2}, \dots, a_{in})$$

$$A = (\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n), \qquad \mathbf{w}_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}$$

Vector algebra

Let $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ be n-dimensional vectors, and $r \in \mathbb{R}$ be a scalar.

Vector sum: $\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$

Scalar multiple: $r\mathbf{a} = (ra_1, ra_2, \ldots, ra_n)$

Zero vector: $\mathbf{0} = (0, 0, \ldots, 0)$

Negative of a vector: $-\mathbf{b} = (-b_1, -b_2, \ldots, -b_n)$

Vector difference:

$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}) = (a_1 - b_1, a_2 - b_2, \ldots, a_n - b_n)$

Given n-dimensional vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ and scalars $r_1, r_2, \ldots, r_k$, the expression

$$\boxed{r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k}$$

is called a **linear combination** of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$.

Also, *vector addition* and *scalar multiplication* are called **linear operations**.

Matrix algebra

Definition. Let $A = (a_{ij})$ and $B = (b_{ij})$ be $m \times n$ matrices. The **sum** $A + B$ is defined to be the $m \times n$ matrix $C = (c_{ij})$ such that $\boxed{c_{ij} = a_{ij} + b_{ij}}$ for all indices i, j.

That is, two matrices with the same dimensions can be added by adding their corresponding entries.

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix} = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \\ a_{31} + b_{31} & a_{32} + b_{32} \end{pmatrix}$$

Definition. Given an $m \times n$ matrix $A = (a_{ij})$ and a number r, the **scalar multiple** rA is defined to be the $m \times n$ matrix $D = (d_{ij})$ such that $\boxed{d_{ij} = ra_{ij}}$ for all indices i, j.

That is, to multiply a matrix by a scalar r, one multiplies each entry of the matrix by r.

$$r \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} ra_{11} & ra_{12} & ra_{13} \\ ra_{21} & ra_{22} & ra_{23} \\ ra_{31} & ra_{32} & ra_{33} \end{pmatrix}$$

The $m \times n$ **zero matrix** (all entries are zeros) is denoted O_{mn} or simply O.

Negative of a matrix: $-A$ is defined as $(-1)A$.

Matrix **difference**: $A - B$ is defined as $A + (-B)$.

As far as the *linear operations* (addition and scalar multiplication) are concerned, the $m \times n$ matrices can be regarded as mn-dimensional vectors.

Examples

$$A = \begin{pmatrix} 3 & 2 & -1 \\ 1 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix},$$

$$C = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

$$A + B = \begin{pmatrix} 5 & 2 & 0 \\ 1 & 2 & 2 \end{pmatrix}, \qquad A - B = \begin{pmatrix} 1 & 2 & -2 \\ 1 & 0 & 0 \end{pmatrix},$$

$$2C = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}, \qquad 3D = \begin{pmatrix} 3 & 3 \\ 0 & 3 \end{pmatrix},$$

$$2C + 3D = \begin{pmatrix} 7 & 3 \\ 0 & 5 \end{pmatrix},$$

$A + D$ is not defined.

Properties of linear operations

$$(A + B) + C = A + (B + C)$$

$$A + B = B + A$$

$$A + O = O + A = A$$

$$A + (-A) = (-A) + A = O$$

$$r(sA) = (rs)A$$

$$r(A + B) = rA + rB$$

$$(r + s)A = rA + sA$$

$$1A = A$$

$$0A = O$$

Dot product

Definition. The **dot product** of n-dimensional vectors $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{y} = (y_1, y_2, \dots, y_n)$ is a scalar

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = \sum_{k=1}^{n} x_k y_k.$$

The dot product is also called the **scalar product**.

Matrix multiplication

The product of matrices A and B is defined if the number of columns in A matches the number of rows in B.

Definition. Let $A = (a_{ik})$ be an $m \times n$ matrix and $B = (b_{kj})$ be an $n \times p$ matrix. The **product** AB is defined to be the $m \times p$ matrix $C = (c_{ij})$ such that $\boxed{c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}}$ for all indices i, j.

That is, matrices are multiplied **row by column**:

$$\begin{pmatrix} * & * & * \\ \boxed{*} & \boxed{*} & \boxed{*} \end{pmatrix} \begin{pmatrix} * & * & \boxed{*} & * \\ * & * & \boxed{*} & * \\ * & * & \boxed{*} & * \end{pmatrix} = \begin{pmatrix} * & * & * & * \\ * & * & \boxed{*} & * \end{pmatrix}$$

$$A = \left(\begin{array}{cccc} a_{11} & a_{12} & \dots & a_{1n} \\ \hline a_{21} & a_{22} & \dots & a_{2n} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline a_{m1} & a_{m2} & \dots & a_{mn} \end{array}\right) = \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_m \end{pmatrix}$$

$$B = \left(\begin{array}{c|c|c|c} b_{11} & b_{12} & \dots & b_{1p} \\ b_{21} & b_{22} & \dots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{np} \end{array}\right) = (\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_p)$$

$$\implies AB = \begin{pmatrix} \mathbf{v}_1 \cdot \mathbf{w}_1 & \mathbf{v}_1 \cdot \mathbf{w}_2 & \dots & \mathbf{v}_1 \cdot \mathbf{w}_p \\ \mathbf{v}_2 \cdot \mathbf{w}_1 & \mathbf{v}_2 \cdot \mathbf{w}_2 & \dots & \mathbf{v}_2 \cdot \mathbf{w}_p \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{v}_m \cdot \mathbf{w}_1 & \mathbf{v}_m \cdot \mathbf{w}_2 & \dots & \mathbf{v}_m \cdot \mathbf{w}_p \end{pmatrix}$$

Examples.

$$(x_1, x_2, \ldots, x_n) \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = (\sum_{k=1}^{n} x_k y_k),$$

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} (x_1, x_2, \ldots, x_n) = \begin{pmatrix} y_1 x_1 & y_1 x_2 & \ldots & y_1 x_n \\ y_2 x_1 & y_2 x_2 & \ldots & y_2 x_n \\ \vdots & \vdots & \ddots & \vdots \\ y_n x_1 & y_n x_2 & \ldots & y_n x_n \end{pmatrix}.$$

Example.

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 3 & 1 & 1 \\ -2 & 5 & 6 & 0 \\ 1 & 7 & 4 & 1 \end{pmatrix} = \begin{pmatrix} -3 & 1 & 3 & 0 \\ -3 & 17 & 16 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 3 & 1 & 1 \\ -2 & 5 & 6 & 0 \\ 1 & 7 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 1 \end{pmatrix} \text{ is not defined}$$

System of linear equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \quad\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

Matrix representation of the system:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \iff A\mathbf{x} = \mathbf{b},$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

Properties of matrix multiplication:

$(AB)C = A(BC)$ (associative law)

$(A + B)C = AC + BC$ (distributive law #1)

$C(A + B) = CA + CB$ (distributive law #2)

$(rA)B = A(rB) = r(AB)$

Any of the above identities holds provided that matrix sums and products are well defined.

If A and B are $n \times n$ matrices, then both AB and BA are well defined $n \times n$ matrices.

However, in general, $AB \neq BA$.

Example. Let $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Then $AB = \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$, $BA = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$.

If AB does equal BA, we say that the matrices A and B **commute**.

Problem. Let A and B be arbitrary $n \times n$ matrices. Is it true that $(A - B)(A + B) = A^2 - B^2$?

$$\begin{aligned} (A - B)(A + B) &= (A - B)A + (A - B)B \\ &= (AA - BA) + (AB - BB) \\ &= A^2 + AB - BA - B^2 \end{aligned}$$

Hence $(A - B)(A + B) = A^2 - B^2$ if and only if A commutes with B.