

MATH 304  
Linear Algebra

**Lecture 7:**  
**Inverse matrix (continued).**

## Identity matrix

*Definition.* The **identity matrix** (or **unit matrix**) is a diagonal matrix with all diagonal entries equal to 1. The  $n \times n$  identity matrix is denoted  $I_n$  or simply  $I$ .

$$I_1 = (1), \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In general, 
$$I = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}.$$

**Theorem.** Let  $A$  be an arbitrary  $m \times n$  matrix. Then  $I_m A = A I_n = A$ .

## Inverse matrix

*Definition.* Let  $A$  be an  $n \times n$  matrix. The **inverse** of  $A$  is an  $n \times n$  matrix, denoted  $A^{-1}$ , such that

$$\boxed{AA^{-1} = A^{-1}A = I.}$$

If  $A^{-1}$  exists then the matrix  $A$  is called **invertible**. Otherwise  $A$  is called **singular**.

Let  $A$  and  $B$  be  $n \times n$  matrices. If  $A$  is invertible then we can **divide**  $B$  by  $A$ :

*left division:*  $A^{-1}B$ , *right division:*  $BA^{-1}$ .

*Remark.* There is no notation for the matrix division and the notion is not used often.

## Basic properties of inverse matrices

- If  $B = A^{-1}$  then  $A = B^{-1}$ . In other words, if  $A$  is invertible, so is  $A^{-1}$ , and  $A = (A^{-1})^{-1}$ .

- The inverse matrix (if it exists) is unique.

Moreover, if  $AB = CA = I$  for some  $n \times n$  matrices  $B$  and  $C$ , then  $B = C = A^{-1}$ .

Indeed,  $B = IB = (CA)B = C(AB) = CI = C$ .

- If  $n \times n$  matrices  $A$  and  $B$  are invertible, so is  $AB$ , and  $(AB)^{-1} = B^{-1}A^{-1}$ .

$$\begin{aligned}(B^{-1}A^{-1})(AB) &= B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I, \\ (AB)(B^{-1}A^{-1}) &= A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I.\end{aligned}$$

- Similarly,  $(A_1A_2 \dots A_k)^{-1} = A_k^{-1} \dots A_2^{-1}A_1^{-1}$ .

## Inverting diagonal matrices

**Theorem** A diagonal matrix  $D = \text{diag}(d_1, \dots, d_n)$  is invertible if and only if all diagonal entries are nonzero:  $d_i \neq 0$  for  $1 \leq i \leq n$ .

If  $D$  is invertible then  $D^{-1} = \text{diag}(d_1^{-1}, \dots, d_n^{-1})$ .

$$\begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{pmatrix}^{-1} = \begin{pmatrix} d_1^{-1} & 0 & \dots & 0 \\ 0 & d_2^{-1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n^{-1} \end{pmatrix}$$

## Inverting $2 \times 2$ matrices

*Definition.* The **determinant** of a  $2 \times 2$  matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ is } \det A = ad - bc.$$

**Theorem** A matrix  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  is invertible if and only if  $\det A \neq 0$ .

If  $\det A \neq 0$  then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

**Problem.** Solve a system  $\begin{cases} 4x + 3y = 5, \\ 3x + 2y = -1. \end{cases}$

This system is equivalent to a matrix equation  $A\mathbf{x} = \mathbf{b}$ ,  
where  $A = \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix}$ ,  $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ ,  $\mathbf{b} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$ .

We have  $\det A = -1 \neq 0$ . Hence  $A$  is invertible.

$$\begin{aligned} A\mathbf{x} = \mathbf{b} &\implies A^{-1}(A\mathbf{x}) = A^{-1}\mathbf{b} \implies (A^{-1}A)\mathbf{x} = A^{-1}\mathbf{b} \\ &\implies \mathbf{x} = A^{-1}\mathbf{b}. \end{aligned}$$

Conversely,  $\mathbf{x} = A^{-1}\mathbf{b} \implies A\mathbf{x} = A(A^{-1}\mathbf{b}) = (AA^{-1})\mathbf{b} = \mathbf{b}$ .

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 5 \\ -1 \end{pmatrix} = \frac{1}{-1} \begin{pmatrix} 2 & -3 \\ -3 & 4 \end{pmatrix} \begin{pmatrix} 5 \\ -1 \end{pmatrix} = \begin{pmatrix} -13 \\ 19 \end{pmatrix}$$

System of  $n$  linear equations in  $n$  variables:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \dots\dots\dots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases} \iff \mathbf{Ax} = \mathbf{b},$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

**Theorem** If the matrix  $A$  is invertible then the system has a unique solution, which is  $\mathbf{x} = A^{-1}\mathbf{b}$ .



## General results on inverse matrices

**Theorem 1** Given a square matrix  $A$ , the following are equivalent:

- (i)  $A$  is invertible;
- (ii)  $\mathbf{x} = \mathbf{0}$  is the only solution of the matrix equation  $A\mathbf{x} = \mathbf{0}$ ;
- (iii) the row echelon form of  $A$  has no zero rows;
- (iv) the reduced row echelon form of  $A$  is the identity matrix.

**Theorem 2** Suppose that a sequence of elementary row operations converts a matrix  $A$  into the identity matrix.

Then the same sequence of operations converts the identity matrix into the inverse matrix  $A^{-1}$ .

**Theorem 3** For any  $n \times n$  matrices  $A$  and  $B$ ,

$$BA = I \iff AB = I.$$

*Row echelon form of a square matrix:*

$$\begin{pmatrix} \square & * & * & * & * & * & * \\ & \square & * & * & * & * & * \\ & & \square & * & * & * & * \\ & & & \square & * & * & * \\ & & & & \square & * & * \\ & & & & & \square & * \\ & & & & & & \square \end{pmatrix}$$

invertible case

$$\begin{pmatrix} \square & * & * & * & * & * & * \\ & \square & * & * & * & * & * \\ & & \square & * & * & * & * \\ & & & \square & * & * & * \\ & & & & \square & * & * \\ & & & & & \square & * \\ & & & & & & \square \end{pmatrix}$$

noninvertible case

*Example.*  $A = \begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix}.$

*To check whether  $A$  is invertible, we convert it to row echelon form.*

Interchange the 1st row with the 2nd row:

$$\begin{pmatrix} 1 & 0 & 1 \\ 3 & -2 & 0 \\ -2 & 3 & 0 \end{pmatrix}$$

Add  $-3$  times the 1st row to the 2nd row:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & -2 & -3 \\ -2 & 3 & 0 \end{pmatrix}$$

Add 2 times the 1st row to the 3rd row:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & -2 & -3 \\ 0 & 3 & 2 \end{pmatrix}$$

Multiply the 2nd row by  $-0.5$ :

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1.5 \\ 0 & 3 & 2 \end{pmatrix}$$

Add  $-3$  times the 2nd row to the 3rd row:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1.5 \\ 0 & 0 & -2.5 \end{pmatrix}$$

Multiply the 3rd row by  $-0.4$ :

$$\begin{pmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & 1.5 \\ 0 & 0 & \boxed{1} \end{pmatrix}$$

*We already know that the matrix  $A$  is invertible.*

*Let's proceed towards reduced row echelon form.*

Add  $-1.5$  times the 3rd row to the 2nd row:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Add  $-1$  times the 3rd row to the 1st row:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

To obtain  $A^{-1}$ , we need to apply the following sequence of elementary row operations to the identity matrix:

- interchange the 1st row with the 2nd row,
- add  $-3$  times the 1st row to the 2nd row,
- add 2 times the 1st row to the 3rd row,
- multiply the 2nd row by  $-0.5$ ,
- add  $-3$  times the 2nd row to the 3rd row,
- multiply the 3rd row by  $-0.4$ ,
- add  $-1.5$  times the 3rd row to the 2nd row,
- add  $-1$  times the 3rd row to the 1st row.

A convenient way to compute the inverse matrix  $A^{-1}$  is to merge the matrices  $A$  and  $I$  into one  $3 \times 6$  matrix  $(A | I)$ , and apply elementary row operations to this new matrix.

$$A = \begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(A | I) = \left( \begin{array}{ccc|ccc} 3 & -2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ -2 & 3 & 0 & 0 & 0 & 1 \end{array} \right)$$

$$\left( \begin{array}{ccc|ccc} 3 & -2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ -2 & 3 & 0 & 0 & 0 & 1 \end{array} \right)$$

Interchange the 1st row with the 2nd row:

$$\left( \begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 3 & -2 & 0 & 1 & 0 & 0 \\ -2 & 3 & 0 & 0 & 0 & 1 \end{array} \right)$$

Add  $-3$  times the 1st row to the 2nd row:

$$\left( \begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -2 & -3 & 1 & -3 & 0 \\ -2 & 3 & 0 & 0 & 0 & 1 \end{array} \right)$$



Add 2 times the 1st row to the 3rd row:

$$\left( \begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -2 & -3 & 1 & -3 & 0 \\ 0 & 3 & 2 & 0 & 2 & 1 \end{array} \right)$$

Multiply the 2nd row by  $-0.5$ :

$$\left( \begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1.5 & -0.5 & 1.5 & 0 \\ 0 & 3 & 2 & 0 & 2 & 1 \end{array} \right)$$

Add  $-3$  times the 2nd row to the 3rd row:

$$\left( \begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1.5 & -0.5 & 1.5 & 0 \\ 0 & 0 & -2.5 & 1.5 & -2.5 & 1 \end{array} \right)$$

Multiply the 3rd row by  $-0.4$ :

$$\left( \begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1.5 & -0.5 & 1.5 & 0 \\ 0 & 0 & 1 & -0.6 & 1 & -0.4 \end{array} \right)$$

Add  $-1.5$  times the 3rd row to the 2nd row:

$$\left( \begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0.4 & 0 & 0.6 \\ 0 & 0 & 1 & -0.6 & 1 & -0.4 \end{array} \right)$$

Add  $-1$  times the 3rd row to the 1st row:

$$\left( \begin{array}{ccc|ccc} 1 & 0 & 0 & 0.6 & 0 & 0.4 \\ 0 & 1 & 0 & 0.4 & 0 & 0.6 \\ 0 & 0 & 1 & -0.6 & 1 & -0.4 \end{array} \right) = (I | A^{-1})$$

Thus 
$$\begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{3}{5} & 0 & \frac{2}{5} \\ \frac{2}{5} & 0 & \frac{3}{5} \\ -\frac{3}{5} & 1 & -\frac{2}{5} \end{pmatrix}.$$

That is,

$$\begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix} \begin{pmatrix} \frac{3}{5} & 0 & \frac{2}{5} \\ \frac{2}{5} & 0 & \frac{3}{5} \\ -\frac{3}{5} & 1 & -\frac{2}{5} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} \frac{3}{5} & 0 & \frac{2}{5} \\ \frac{2}{5} & 0 & \frac{3}{5} \\ -\frac{3}{5} & 1 & -\frac{2}{5} \end{pmatrix} \begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$