

MATH 304
Linear Algebra

Lecture 15:
Wronskian.
Basis of a vector space.

Linear independence

Definition. Let V be a vector space. Vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in V$ are called **linearly dependent** if they satisfy a relation

$$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k = \mathbf{0},$$

where the coefficients $r_1, \dots, r_k \in \mathbb{R}$ are not all equal to zero. Otherwise the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are called **linearly independent**. That is, if

$$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k = \mathbf{0} \implies r_1 = \cdots = r_k = 0.$$

An infinite set $S \subset V$ is **linearly dependent** if there are some linearly dependent vectors $\mathbf{v}_1, \dots, \mathbf{v}_k \in S$. Otherwise S is **linearly independent**.

Remark. If a set S (finite or infinite) is linearly independent then any subset of S is also linearly independent.

Theorem Vectors $\mathbf{v}_1, \dots, \mathbf{v}_k \in V$ are linearly dependent if and only if one of them is a linear combination of the other $k - 1$ vectors.

Examples of linear independence.

- Vectors $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, and $\mathbf{e}_3 = (0, 0, 1)$ in \mathbb{R}^3 .

- Matrices $E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$,

$E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and $E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

- Polynomials $1, x, x^2, \dots, x^n, \dots$

Problem. Show that functions e^x , e^{2x} , and e^{3x} are linearly independent in $C^\infty(\mathbb{R})$.

Suppose that $ae^x + be^{2x} + ce^{3x} = 0$ for all $x \in \mathbb{R}$, where a, b, c are constants. We have to show that $a = b = c = 0$.

Differentiate this identity twice:

$$\begin{aligned}ae^x + be^{2x} + ce^{3x} &= 0, \\ae^x + 2be^{2x} + 3ce^{3x} &= 0, \\ae^x + 4be^{2x} + 9ce^{3x} &= 0.\end{aligned}$$

It follows that $A(x)\mathbf{v} = \mathbf{0}$, where

$$A(x) = \begin{pmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

$$A(x) = \begin{pmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

$$\begin{aligned} \det A(x) &= e^x \begin{vmatrix} 1 & e^{2x} & e^{3x} \\ 1 & 2e^{2x} & 3e^{3x} \\ 1 & 4e^{2x} & 9e^{3x} \end{vmatrix} = e^x e^{2x} \begin{vmatrix} 1 & 1 & e^{3x} \\ 1 & 2 & 3e^{3x} \\ 1 & 4 & 9e^{3x} \end{vmatrix} \\ &= e^x e^{2x} e^{3x} \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = e^{6x} \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = e^{6x} \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 4 & 9 \end{vmatrix} \\ &= e^{6x} \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 8 \end{vmatrix} = e^{6x} \begin{vmatrix} 1 & 2 \\ 3 & 8 \end{vmatrix} = 2e^{6x} \neq 0. \end{aligned}$$

Since the matrix $A(x)$ is invertible, we obtain

$$A(x)\mathbf{v} = \mathbf{0} \implies \mathbf{v} = \mathbf{0} \implies a = b = c = 0$$

Wronskian

Let f_1, f_2, \dots, f_n be smooth functions on an interval $[a, b]$. The **Wronskian** $W[f_1, f_2, \dots, f_n]$ is a function on $[a, b]$ defined by

$$W[f_1, f_2, \dots, f_n](x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}.$$

Theorem If $W[f_1, f_2, \dots, f_n](x_0) \neq 0$ for some $x_0 \in [a, b]$ then the functions f_1, f_2, \dots, f_n are linearly independent in $C[a, b]$.

Theorem 1 Let $\lambda_1, \lambda_2, \dots, \lambda_k$ be distinct real numbers. Then the functions $e^{\lambda_1 x}, e^{\lambda_2 x}, \dots, e^{\lambda_k x}$ are linearly independent.

Theorem 2 The set of functions

$$\{x^m e^{\lambda x} \mid \lambda \in \mathbb{R}, m = 0, 1, 2, \dots\}$$

is linearly independent.

Spanning set

Let S be a subset of a vector space V .

Definition. The **span** of the set S is the smallest subspace $W \subset V$ that contains S . If S is not empty then $W = \text{Span}(S)$ consists of all linear combinations $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k$ such that $\mathbf{v}_1, \dots, \mathbf{v}_k \in S$ and $r_1, \dots, r_k \in \mathbb{R}$.

We say that the set S **spans** the subspace W or that S is a **spanning set** for W .

Remark. If S_1 is a spanning set for a vector space V and $S_1 \subset S_2 \subset V$, then S_2 is also a spanning set for V .

Basis

Definition. Let V be a vector space. A linearly independent spanning set for V is called a **basis**.

Suppose that a set $S \subset V$ is a basis for V .

“Spanning set” means that any vector $\mathbf{v} \in V$ can be represented as a linear combination

$$\mathbf{v} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k,$$

where $\mathbf{v}_1, \dots, \mathbf{v}_k$ are distinct vectors from S and $r_1, \dots, r_k \in \mathbb{R}$. “Linearly independent” implies that the above representation is unique:

$$\mathbf{v} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k = r'_1\mathbf{v}_1 + r'_2\mathbf{v}_2 + \cdots + r'_k\mathbf{v}_k$$

$$\implies (r_1 - r'_1)\mathbf{v}_1 + (r_2 - r'_2)\mathbf{v}_2 + \cdots + (r_k - r'_k)\mathbf{v}_k = \mathbf{0}$$

$$\implies r_1 - r'_1 = r_2 - r'_2 = \cdots = r_k - r'_k = 0$$

Examples. • Standard basis for \mathbb{R}^n :

$$\mathbf{e}_1 = (1, 0, 0, \dots, 0, 0), \quad \mathbf{e}_2 = (0, 1, 0, \dots, 0, 0), \dots, \\ \mathbf{e}_n = (0, 0, 0, \dots, 0, 1).$$

Indeed, $(x_1, x_2, \dots, x_n) = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \dots + x_n\mathbf{e}_n$.

- Matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

form a basis for $\mathcal{M}_{2,2}(\mathbb{R})$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

- Polynomials $1, x, x^2, \dots, x^{n-1}$ form a basis for $\mathcal{P}_n = \{a_0 + a_1x + \dots + a_{n-1}x^{n-1} : a_i \in \mathbb{R}\}$.

- The infinite set $\{1, x, x^2, \dots, x^n, \dots\}$ is a basis for \mathcal{P} , the space of all polynomials.

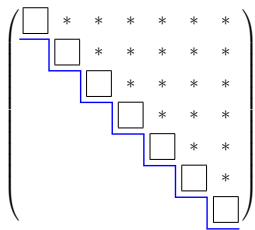
Let $\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in \mathbb{R}^n$ and $r_1, r_2, \dots, r_k \in \mathbb{R}$.

The vector equation $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \dots + r_k\mathbf{v}_k = \mathbf{v}$ is equivalent to the matrix equation $A\mathbf{x} = \mathbf{v}$, where

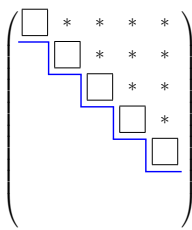
$$A = (\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k), \quad \mathbf{x} = \begin{pmatrix} r_1 \\ \vdots \\ r_k \end{pmatrix}.$$

That is, A is the $n \times k$ matrix such that vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are consecutive columns of A .

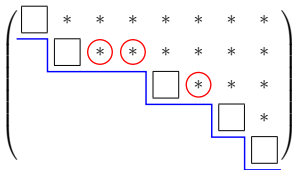
- *Vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ span \mathbb{R}^n if the row echelon form of A has no zero rows.*
- *Vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ are linearly independent if the row echelon form of A has a leading entry in each column (no free variables).*



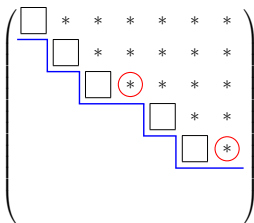
spanning
linear independence



no spanning
linear independence



spanning
no linear independence



no spanning
no linear independence

Bases for \mathbb{R}^n

Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ be vectors in \mathbb{R}^n .

Theorem 1 If $k < n$ then the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ do not span \mathbb{R}^n .

Theorem 2 If $k > n$ then the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are linearly dependent.

Theorem 3 If $k = n$ then the following conditions are equivalent:

- (i) $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for \mathbb{R}^n ;
- (ii) $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a spanning set for \mathbb{R}^n ;
- (iii) $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a linearly independent set.

Example. Consider vectors $\mathbf{v}_1 = (1, -1, 1)$, $\mathbf{v}_2 = (1, 0, 0)$, $\mathbf{v}_3 = (1, 1, 1)$, and $\mathbf{v}_4 = (1, 2, 4)$ in \mathbb{R}^3 .

Vectors \mathbf{v}_1 and \mathbf{v}_2 are linearly independent (as they are not parallel), but they do not span \mathbb{R}^3 .

Vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent since

$$\begin{vmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{vmatrix} = - \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -(-2) = 2 \neq 0.$$

Therefore $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for \mathbb{R}^3 .

Vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ span \mathbb{R}^3 (because $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ already span \mathbb{R}^3), but they are linearly dependent.