MATH 304

Linear Algebra

Lecture 20: Linear transformations. Range and kernel.

Linear mapping = linear transformation = linear function

Definition. Given vector spaces V_1 and V_2, a mapping $L : V_1 \to V_2$ is **linear** if

$$\boxed{L(\mathbf{x}+\mathbf{y}) = L(\mathbf{x}) + L(\mathbf{y}),}$$

$$\boxed{L(r\mathbf{x}) = rL(\mathbf{x})}$$

for any $\mathbf{x}, \mathbf{y} \in V_1$ and $r \in \mathbb{R}$.

A linear mapping $\ell : V \to \mathbb{R}$ is called a **linear functional** on V.

If $V_1 = V_2$ (or if both V_1 and V_2 are functional spaces) then a linear mapping $L : V_1 \to V_2$ is called a **linear operator**.

Linear mapping = linear transformation = linear function

Definition. Given vector spaces V_1 and V_2, a mapping $L : V_1 \to V_2$ is **linear** if

$$\boxed{L(\mathbf{x}+\mathbf{y}) = L(\mathbf{x}) + L(\mathbf{y}),}$$

$$\boxed{L(r\mathbf{x}) = rL(\mathbf{x})}$$

for any $\mathbf{x}, \mathbf{y} \in V_1$ and $r \in \mathbb{R}$.

Remark. A function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = ax + b$ is a linear transformation of the vector space $\mathbb{R}$ if and only if $b = 0$.

Properties of linear mappings

Let $L : V_1 \to V_2$ be a linear mapping.

- $L(r_1\mathbf{v}_1 + \cdots + r_k\mathbf{v}_k) = r_1 L(\mathbf{v}_1) + \cdots + r_k L(\mathbf{v}_k)$

for all $k \geq 1$, $\mathbf{v}_1, \dots, \mathbf{v}_k \in V_1$, and $r_1, \dots, r_k \in \mathbb{R}$.

$L(r_1\mathbf{v}_1 + r_2\mathbf{v}_2) = L(r_1\mathbf{v}_1) + L(r_2\mathbf{v}_2) = r_1 L(\mathbf{v}_1) + r_2 L(\mathbf{v}_2)$,
$L(r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + r_3\mathbf{v}_3) = L(r_1\mathbf{v}_1 + r_2\mathbf{v}_2) + L(r_3\mathbf{v}_3) =$
$= r_1 L(\mathbf{v}_1) + r_2 L(\mathbf{v}_2) + r_3 L(\mathbf{v}_3)$, and so on.

- $L(\mathbf{0}_1) = \mathbf{0}_2$, where $\mathbf{0}_1$ and $\mathbf{0}_2$ are zero vectors in

V_1 and V_2, respectively.

$L(\mathbf{0}_1) = L(0\mathbf{0}_1) = 0L(\mathbf{0}_1) = \mathbf{0}_2$.

- $L(-\mathbf{v}) = -L(\mathbf{v})$ for any $\mathbf{v} \in V_1$.

$L(-\mathbf{v}) = L((-1)\mathbf{v}) = (-1)L(\mathbf{v}) = -L(\mathbf{v})$.

Examples of linear mappings

- *Scaling* $L : V \to V,\ L(\mathbf{v}) = s\mathbf{v},$ where $s \in \mathbb{R}.$

$L(\mathbf{x}+\mathbf{y}) = s(\mathbf{x}+\mathbf{y}) = s\mathbf{x} + s\mathbf{y} = L(\mathbf{x}) + L(\mathbf{y}),$
$L(r\mathbf{x}) = s(r\mathbf{x}) = r(s\mathbf{x}) = rL(\mathbf{x}).$

- *Dot product with a fixed vector*

$\ell : \mathbb{R}^n \to \mathbb{R},\ \ell(\mathbf{v}) = \mathbf{v} \cdot \mathbf{v}_0,$ where $\mathbf{v}_0 \in \mathbb{R}^n.$

$\ell(\mathbf{x}+\mathbf{y}) = (\mathbf{x}+\mathbf{y}) \cdot \mathbf{v}_0 = \mathbf{x} \cdot \mathbf{v}_0 + \mathbf{y} \cdot \mathbf{v}_0 = \ell(\mathbf{x}) + \ell(\mathbf{y}),$
$\ell(r\mathbf{x}) = (r\mathbf{x}) \cdot \mathbf{v}_0 = r(\mathbf{x} \cdot \mathbf{v}_0) = r\ell(\mathbf{x}).$

- *Cross product with a fixed vector*

$L : \mathbb{R}^3 \to \mathbb{R}^3,\ L(\mathbf{v}) = \mathbf{v} \times \mathbf{v}_0,$ where $\mathbf{v}_0 \in \mathbb{R}^3.$

- *Multiplication by a fixed matrix*

$L : \mathbb{R}^n \to \mathbb{R}^m,\ L(\mathbf{v}) = A\mathbf{v},$ where A is an $m \times n$ matrix and all vectors are column vectors.

Linear mappings of functional vector spaces

- *Evaluation at a fixed point*

$\ell : F(\mathbb{R}) \to \mathbb{R}$, $\ell(f) = f(a)$, where $a \in \mathbb{R}$.

- *Multiplication by a fixed function*

$L : F(\mathbb{R}) \to F(\mathbb{R})$, $L(f) = gf$, where $g \in F(\mathbb{R})$.

- *Differentiation* $D : C^1(\mathbb{R}) \to C(\mathbb{R})$, $L(f) = f'$.

$D(f + g) = (f + g)' = f' + g' = D(f) + D(g)$,
$D(rf) = (rf)' = rf' = rD(f)$.

- *Integration over a finite interval*

$\ell : C(\mathbb{R}) \to \mathbb{R}$, $\ell(f) = \int_a^b f(x)\,dx$, where
$a, b \in \mathbb{R}$, $a < b$.

Properties of linear mappings

- If a linear mapping $L : V \longrightarrow W$ is invertible then the inverse mapping $L^{-1} : W \longrightarrow V$ is also linear.

- If $L : V \longrightarrow W$ and $M : W \longrightarrow X$ are linear mappings then the composition $M \circ L : V \longrightarrow X$ is also linear.

- If $L_1 : V \longrightarrow W$ and $L_2 : V \longrightarrow W$ are linear mappings then the sum $L_1 + L_2$ is also linear.

Linear differential operators

- an ordinary differential operator

$$L : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}), \qquad L = g_0 \frac{d^2}{dx^2} + g_1 \frac{d}{dx} + g_2,$$

where g_0, g_1, g_2 are smooth functions on $\mathbb{R}$.
That is, $L(f) = g_0 f'' + g_1 f' + g_2 f$.

- Laplace's operator $\Delta : C^\infty(\mathbb{R}^2) \to C^\infty(\mathbb{R}^2)$,

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

(a.k.a. the Laplacian; also denoted by ∇^2).

Range and kernel

Let V, W be vector spaces and $L : V \to W$ be a linear mapping.

Definition. The **range** (or **image**) of L is the set of all vectors $\mathbf{w} \in W$ such that $\mathbf{w} = L(\mathbf{v})$ for some $\mathbf{v} \in V$. The range of L is denoted $L(V)$.

The **kernel** of L, denoted $\ker L$, is the set of all vectors $\mathbf{v} \in V$ such that $L(\mathbf{v}) = \mathbf{0}$.

Theorem **(i)** The range of L is a subspace of W.
(ii) The kernel of L is a subspace of V.

Example. $L:\mathbb{R}^3\to\mathbb{R}^3$, $L\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1&0&-1\\1&2&-1\\1&0&-1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$.

The kernel $\ker L$ is the nullspace of the matrix.

$$L\begin{pmatrix}x\\y\\z\end{pmatrix}=x\begin{pmatrix}1\\1\\1\end{pmatrix}+y\begin{pmatrix}0\\2\\0\end{pmatrix}+z\begin{pmatrix}-1\\-1\\-1\end{pmatrix}$$

The range $f(\mathbb{R}^3)$ is the column space of the matrix.

Example. $L : \mathbb{R}^3 \to \mathbb{R}^3$, $L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & -1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

The range of L is spanned by vectors $(1, 1, 1)$, $(0, 2, 0)$, and $(-1, -1, -1)$. It follows that $L(\mathbb{R}^3)$ is the plane spanned by $(1, 1, 1)$ and $(0, 1, 0)$.

To find $\ker L$, we apply row reduction to the matrix:

$$\begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & -1 \\ 1 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence $(x, y, z) \in \ker L$ if $x - z = y = 0$.
It follows that $\ker L$ is the line spanned by $(1, 0, 1)$.

More examples

$$f : \mathcal{M}_2(\mathbb{R}) \to \mathcal{M}_2(\mathbb{R}), \;\; f(A) = A + A^T.$$

$$f\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 2a & b+c \\ b+c & 2d \end{pmatrix}.$$

$\ker f$ is the subspace of anti-symmetric matrices, the range of f is the subspace of symmetric matrices.

$$g : \mathcal{M}_2(\mathbb{R}) \to \mathcal{M}_2(\mathbb{R}), \;\; g(A) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} A.$$

$$g\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix}.$$

The range of g is the subspace of matrices with the zero second row, $\ker g$ is the same as the range $\implies g(g(A)) = O$.

$\mathcal{P}$: the space of polynomials.
$\mathcal{P}_n$: the space of polynomials of degree less than n.

$D : \mathcal{P} \to \mathcal{P}$, $(Dp)(x) = p'(x)$.
$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n$
$\implies (Dp)(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1}$

The range of D is the entire $\mathcal{P}$, $\ker D = \mathcal{P}_1 =$ the subspace of constants.

$D : \mathcal{P}_4 \to \mathcal{P}_4$, $(Dp)(x) = p'(x)$.

$$p(x) = ax^3 + bx^2 + cx + d \implies (Dp)(x) = 3ax^2 + 2bx + c$$

The range of D is $\mathcal{P}_3$, $\ker D = \mathcal{P}_1$.