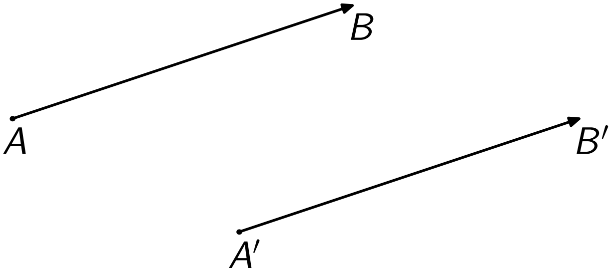
MATH 304

Linear Algebra

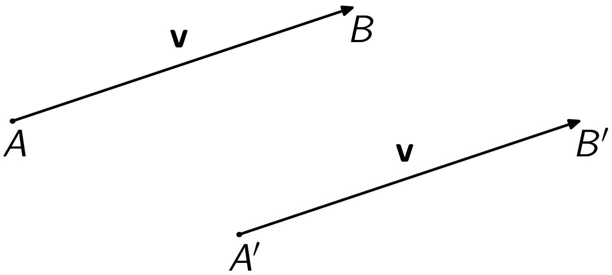
Lecture 23: Euclidean structure in $\mathbb{R}^n$. Orthogonality.

Vectors: geometric approach



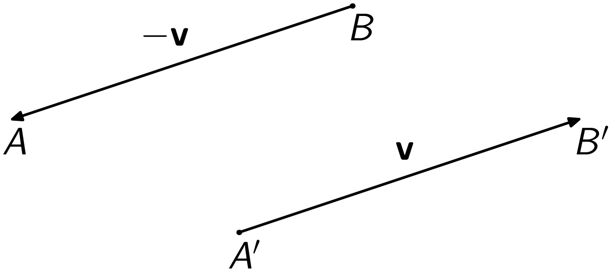
- A vector is represented by a directed segment.
- Directed segment is drawn as an arrow.
- Different arrows represent the same vector if they are of the same length and direction.

Vectors: geometric approach



$\overrightarrow{AB}$ denotes the vector represented by the arrow with tip at B and tail at A.
$\overrightarrow{AA}$ is called the *zero vector* and denoted $\mathbf{0}$.

Vectors: geometric approach

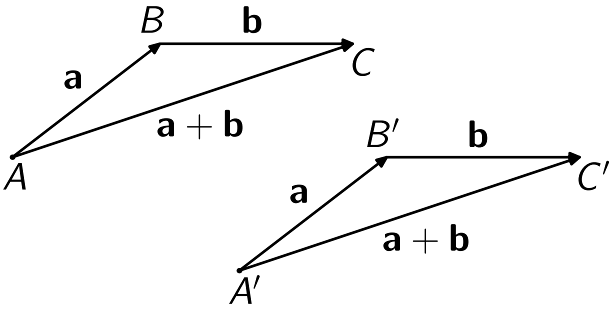


If $\mathbf{v} = \overrightarrow{AB}$ then $\overrightarrow{BA}$ is called the *negative vector* of $\mathbf{v}$ and denoted $-\mathbf{v}$.

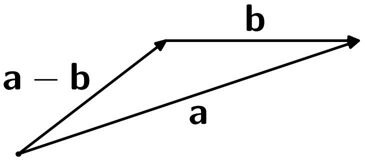
Linear structure: vector addition

Given vectors **a** and **b**, their *sum* $\mathbf{a}+\mathbf{b}$ is defined by the rule $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$.

That is, choose points A, B, C so that $\overrightarrow{AB}=\mathbf{a}$ and $\overrightarrow{BC}=\mathbf{b}$. Then $\mathbf{a}+\mathbf{b}=\overrightarrow{AC}$.

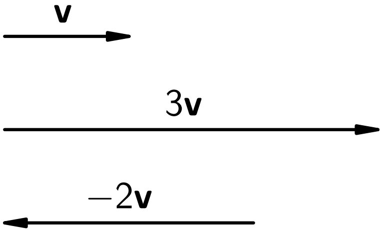


The *difference* of the two vectors is defined as
$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$.



Linear structure: scalar multiplication

Let $\mathbf{v}$ be a vector and $r \in \mathbb{R}$. By definition, $r\mathbf{v}$ is a vector whose magnitude is $|r|$ times the magnitude of $\mathbf{v}$. The direction of $r\mathbf{v}$ coincides with that of $\mathbf{v}$ if $r > 0$. If $r < 0$ then the directions of $r\mathbf{v}$ and $\mathbf{v}$ are opposite.



Beyond linearity: length of a vector

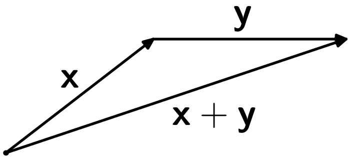
The **length** (or the **magnitude**) of a vector $\overrightarrow{AB}$ is the length of the representing segment AB. The length of a vector $\mathbf{v}$ is denoted $|\mathbf{v}|$ or $\|\mathbf{v}\|$.

Properties of vector length:

$|\mathbf{x}| \geq 0$, $|\mathbf{x}| = 0$ only if $\mathbf{x} = \mathbf{0}$ (positivity)

$|r\mathbf{x}| = |r|\,|\mathbf{x}|$ (homogeneity)

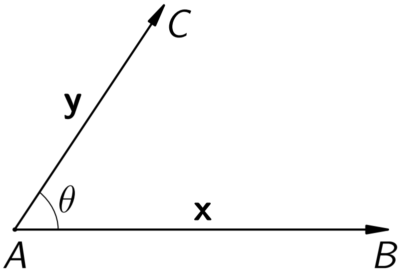
$|\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$ (triangle inequality)

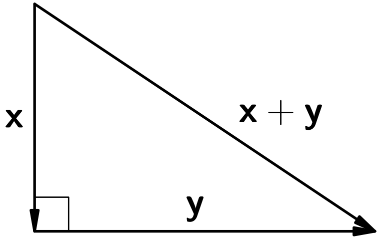


Beyond linearity: angle between vectors

Given nonzero vectors $\mathbf{x}$ and $\mathbf{y}$, let A, B, and C be points such that $\overrightarrow{AB} = \mathbf{x}$ and $\overrightarrow{AC} = \mathbf{y}$. Then $\angle BAC$ is called the **angle** between $\mathbf{x}$ and $\mathbf{y}$.

The vectors $\mathbf{x}$ and $\mathbf{y}$ are called **orthogonal** (denoted $\mathbf{x} \perp \mathbf{y}$) if the angle between them equals $90^{\rm o}$.





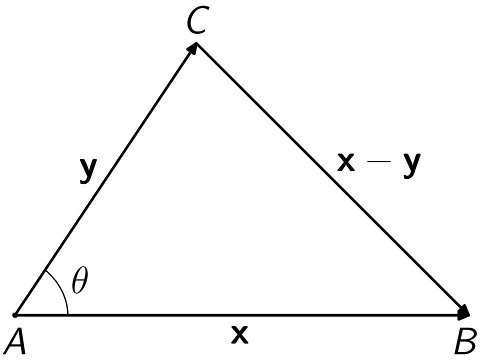
Pythagorean Theorem:

$$\mathbf{x} \perp \mathbf{y} \implies |\mathbf{x} + \mathbf{y}|^2 = |\mathbf{x}|^2 + |\mathbf{y}|^2$$

3-*dimensional Pythagorean Theorem:*

If vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are pairwise orthogonal then

$$|\mathbf{x} + \mathbf{y} + \mathbf{z}|^2 = |\mathbf{x}|^2 + |\mathbf{y}|^2 + |\mathbf{z}|^2$$



Law of cosines:

$$|\mathbf{x}-\mathbf{y}|^2 = |\mathbf{x}|^2 + |\mathbf{y}|^2 - 2|\mathbf{x}|\,|\mathbf{y}|\,\cos\theta$$

Beyond linearity: dot product

The **dot product** of vectors $\mathbf{x}$ and $\mathbf{y}$ is

$$\mathbf{x} \cdot \mathbf{y} = |\mathbf{x}|\,|\mathbf{y}|\,\cos\theta,$$

where θ is the angle between $\mathbf{x}$ and $\mathbf{y}$.

The dot product is also called the **scalar product**.
Alternative notation: $(\mathbf{x}, \mathbf{y})$ or $\langle \mathbf{x}, \mathbf{y} \rangle$.

The vectors $\mathbf{x}$ and $\mathbf{y}$ are orthogonal if and only if
$\mathbf{x} \cdot \mathbf{y} = 0$.

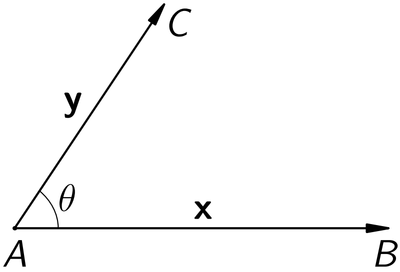
Relations between lengths and dot products:

- $|\mathbf{x}| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$
- $|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}|\,|\mathbf{y}|$
- $|\mathbf{x} - \mathbf{y}|^2 = |\mathbf{x}|^2 + |\mathbf{y}|^2 - 2\,\mathbf{x}\cdot\mathbf{y}$

Euclidean structure

Euclidean structure includes:

- length of a vector: $|\mathbf{x}|$,
- angle between vectors: θ,
- dot product: $\mathbf{x} \cdot \mathbf{y} = |\mathbf{x}|\,|\mathbf{y}|\,\cos\theta$.



Vectors: algebraic approach

An n-dimensional coordinate vector is an element of $\mathbb{R}^n$, i.e., an ordered n-tuple $(x_1, x_2, \ldots, x_n)$ of real numbers.

Let $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ be vectors, and $r \in \mathbb{R}$ be a scalar. Then, by definition,

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n),$$

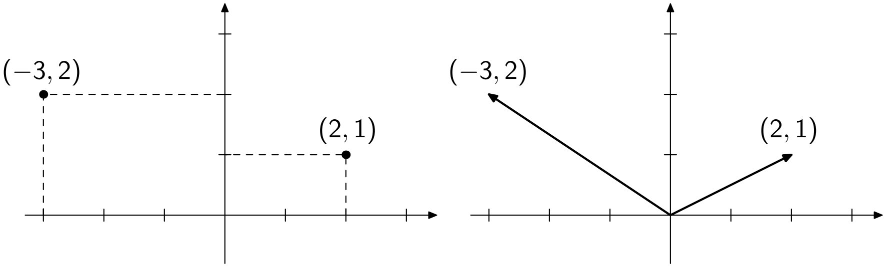
$$r\mathbf{a} = (ra_1, ra_2, \ldots, ra_n),$$

$$\mathbf{0} = (0, 0, \ldots, 0),$$

$$-\mathbf{b} = (-b_1, -b_2, \ldots, -b_n),$$

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}) = (a_1 - b_1, a_2 - b_2, \ldots, a_n - b_n).$$

Cartesian coordinates: geometric meets algebraic



Once we specify an *origin* O, each point A is associated a *position vector* $\overrightarrow{OA}$. Conversely, every vector has a unique representative with tail at O.

Cartesian coordinates allow us to identify a line, a plane, and space with $\mathbb{R}$, $\mathbb{R}^2$, and $\mathbb{R}^3$, respectively.

Length and distance

Definition. The **length** of a vector $\mathbf{v} = (v_1, v_2, \dots, v_n) \in \mathbb{R}^n$ is

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

The **distance** between vectors/points $\mathbf{x}$ and $\mathbf{y}$ is $\|\mathbf{y} - \mathbf{x}\|$.

Properties of length:

$\|\mathbf{x}\| \geq 0$, $\|\mathbf{x}\| = 0$ only if $\mathbf{x} = \mathbf{0}$ (positivity)

$\|r\mathbf{x}\| = |r| \, \|\mathbf{x}\|$ (homogeneity)

$\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ (triangle inequality)

Scalar product

Definition. The **scalar product** of vectors $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{y} = (y_1, y_2, \dots, y_n)$ is

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = \sum_{k=1}^{n} x_k y_k .$$

Properties of scalar product:

$\mathbf{x} \cdot \mathbf{x} \geq 0$, $\mathbf{x} \cdot \mathbf{x} = 0$ only if $\mathbf{x} = \mathbf{0}$ (positivity)

$\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$ (symmetry)

$(\mathbf{x} + \mathbf{y}) \cdot \mathbf{z} = \mathbf{x} \cdot \mathbf{z} + \mathbf{y} \cdot \mathbf{z}$ (distributive law)

$(r\mathbf{x}) \cdot \mathbf{y} = r(\mathbf{x} \cdot \mathbf{y})$ (homogeneity)

Relations between lengths and scalar products:

$\|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$

$|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\| \, \|\mathbf{y}\|$ (Cauchy-Schwarz inequality)

$\|\mathbf{x} - \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2\, \mathbf{x} \cdot \mathbf{y}$

By the Cauchy-Schwarz inequality, for any nonzero vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ we have

$$\cos\theta = \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\| \, \|\mathbf{y}\|} \quad \text{for some} \quad 0 \leq \theta \leq \pi.$$

θ is called the **angle** between the vectors **x** and **y**. The vectors **x** and **y** are said to be **orthogonal** (denoted $\mathbf{x} \perp \mathbf{y}$) if $\mathbf{x} \cdot \mathbf{y} = 0$ (i.e., if $\theta = 90^{\circ}$).

Problem. Find the angle θ between vectors $\mathbf{x} = (2, -1)$ and $\mathbf{y} = (3, 1)$.

$\mathbf{x} \cdot \mathbf{y} = 5$, $\|\mathbf{x}\| = \sqrt{5}$, $\|\mathbf{y}\| = \sqrt{10}$.

$$\cos\theta = \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\|\,\|\mathbf{y}\|} = \frac{5}{\sqrt{5}\,\sqrt{10}} = \frac{1}{\sqrt{2}} \implies \theta = 45^{\circ}$$

Problem. Find the angle ϕ between vectors $\mathbf{v} = (-2, 1, 3)$ and $\mathbf{w} = (4, 5, 1)$.

$$\mathbf{v} \cdot \mathbf{w} = 0 \implies \mathbf{v} \perp \mathbf{w} \implies \phi = 90^{\circ}$$

Orthogonality

Definition 1. Vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ are said to be **orthogonal** (denoted $\mathbf{x} \perp \mathbf{y}$) if $\boxed{\mathbf{x} \cdot \mathbf{y} = 0.}$

Definition 2. A vector $\mathbf{x} \in \mathbb{R}^n$ is said to be **orthogonal** to a nonempty set $Y \subset \mathbb{R}^n$ (denoted $\mathbf{x} \perp Y$) if $\mathbf{x} \cdot \mathbf{y} = 0$ for any $\mathbf{y} \in Y$.

Definition 3. Nonempty sets $X, Y \subset \mathbb{R}^n$ are said to be **orthogonal** (denoted $X \perp Y$) if $\mathbf{x} \cdot \mathbf{y} = 0$ for any $\mathbf{x} \in X$ and $\mathbf{y} \in Y$.

Examples in $\mathbb{R}^3$. • The line $x = y = 0$ is orthogonal to the line $y = z = 0$.

Indeed, if $\mathbf{v} = (0, 0, z)$ and $\mathbf{w} = (x, 0, 0)$ then $\mathbf{v} \cdot \mathbf{w} = 0$.

• The line $x = y = 0$ is orthogonal to the plane $z = 0$.

Indeed, if $\mathbf{v} = (0, 0, z)$ and $\mathbf{w} = (x, y, 0)$ then $\mathbf{v} \cdot \mathbf{w} = 0$.

• The line $x = y = 0$ is not orthogonal to the plane $z = 1$.

The vector $\mathbf{v} = (0, 0, 1)$ belongs to both the line and the plane, and $\mathbf{v} \cdot \mathbf{v} = 1 \neq 0$.

• The plane $z = 0$ is not orthogonal to the plane $y = 0$.

The vector $\mathbf{v} = (1, 0, 0)$ belongs to both planes and $\mathbf{v} \cdot \mathbf{v} = 1 \neq 0$.