MATH 304

Linear Algebra

Lecture 28:
Orthogonal bases.
The Gram-Schmidt orthogonalization process.

Orthogonal sets

Let V be an inner product space with an inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$.

Definition. A nonempty set $S \subset V$ of nonzero vectors is called an **orthogonal set** if all vectors in S are mutually orthogonal. That is, $\mathbf{0} \notin S$ and $\langle \mathbf{x}, \mathbf{y} \rangle = 0$ for any $\mathbf{x}, \mathbf{y} \in S$, $\mathbf{x} \neq \mathbf{y}$.

An orthogonal set $S \subset V$ is called **orthonormal** if $\|\mathbf{x}\| = 1$ for any $\mathbf{x} \in S$.

Remark. Vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in V$ form an orthonormal set if and only if

$$\langle \mathbf{v}_i, \mathbf{v}_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Examples. • $V = \mathbb{R}^n$, $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y}$.

The standard basis $\mathbf{e}_1 = (1, 0, 0, \ldots, 0)$, $\mathbf{e}_2 = (0, 1, 0, \ldots, 0)$, $\ldots$, $\mathbf{e}_n = (0, 0, 0, \ldots, 1)$. It is an orthonormal set.

• $V = \mathbb{R}^3$, $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y}$.

$\mathbf{v}_1 = (3, 5, 4)$, $\mathbf{v}_2 = (3, -5, 4)$, $\mathbf{v}_3 = (4, 0, -3)$.

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = 0, \quad \mathbf{v}_1 \cdot \mathbf{v}_3 = 0, \quad \mathbf{v}_2 \cdot \mathbf{v}_3 = 0,$$
$$\mathbf{v}_1 \cdot \mathbf{v}_1 = 50, \quad \mathbf{v}_2 \cdot \mathbf{v}_2 = 50, \quad \mathbf{v}_3 \cdot \mathbf{v}_3 = 25.$$

Thus the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is orthogonal but not orthonormal. An orthonormal set is formed by normalized vectors $\mathbf{w}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}$, $\mathbf{w}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|}$, $\mathbf{w}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|}$.

- $V = C[-\pi, \pi]$, $\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x)\,dx.$

$f_1(x) = \sin x$, $f_2(x) = \sin 2x$, $\ldots$, $f_n(x) = \sin nx$, $\ldots$

$$\langle f_m, f_n \rangle = \int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx = \begin{cases} \pi & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

Thus the set $\{f_1, f_2, f_3, \ldots\}$ is orthogonal but not orthonormal.

It is orthonormal with respect to a scaled inner product

$$\langle\langle f, g \rangle\rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x)\,dx.$$

Orthogonality $\Longrightarrow$ linear independence

Theorem Suppose $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are nonzero vectors that form an orthogonal set. Then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent.

Proof: Suppose $t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_k\mathbf{v}_k = \mathbf{0}$ for some $t_1, t_2, \ldots, t_k \in \mathbb{R}$.

Then for any index $1 \leq i \leq k$ we have

$$\langle t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_k\mathbf{v}_k, \mathbf{v}_i \rangle = \langle \mathbf{0}, \mathbf{v}_i \rangle = 0.$$

$$\Longrightarrow \; t_1\langle \mathbf{v}_1, \mathbf{v}_i \rangle + t_2\langle \mathbf{v}_2, \mathbf{v}_i \rangle + \cdots + t_k\langle \mathbf{v}_k, \mathbf{v}_i \rangle = 0$$

By orthogonality, $t_i\langle \mathbf{v}_i, \mathbf{v}_i \rangle = 0 \Longrightarrow t_i = 0$.

Orthonormal bases

Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ be an orthonormal basis for an inner product space V.

Theorem Let $\mathbf{x} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n$ and $\mathbf{y} = y_1\mathbf{v}_1 + y_2\mathbf{v}_2 + \cdots + y_n\mathbf{v}_n$, where $x_i, y_j \in \mathbb{R}$. Then

(i) $\langle \mathbf{x}, \mathbf{y} \rangle = x_1y_1 + x_2y_2 + \cdots + x_ny_n$,

(ii) $\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$.

Proof: (ii) follows from (i) when $\mathbf{y} = \mathbf{x}$.

$$\langle \mathbf{x}, \mathbf{y} \rangle = \left\langle \sum_{i=1}^n x_i\mathbf{v}_i, \sum_{j=1}^n y_j\mathbf{v}_j \right\rangle = \sum_{i=1}^n x_i \left\langle \mathbf{v}_i, \sum_{j=1}^n y_j\mathbf{v}_j \right\rangle$$

$$= \sum_{i=1}^n \sum_{j=1}^n x_i y_j \langle \mathbf{v}_i, \mathbf{v}_j \rangle = \sum_{i=1}^n x_i y_i.$$

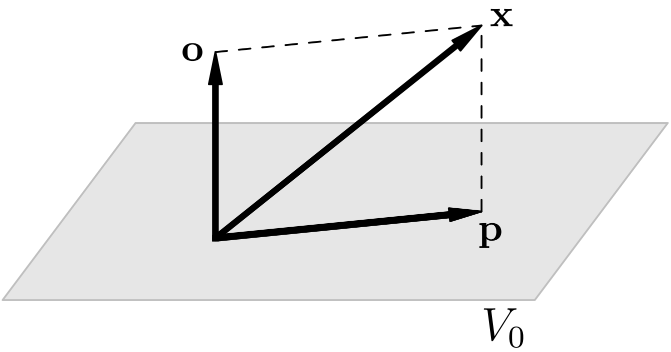
Orthogonal projection

Theorem Let V be an inner product space and V_0 be a finite-dimensional subspace of V. Then any vector $\mathbf{x} \in V$ is uniquely represented as $\mathbf{x} = \mathbf{p} + \mathbf{o}$, where $\mathbf{p} \in V_0$ and $\mathbf{o} \perp V_0$.

The component $\mathbf{p}$ is the **orthogonal projection** of the vector $\mathbf{x}$ onto the subspace V_0. We have

$$\|\mathbf{o}\| = \|\mathbf{x} - \mathbf{p}\| = \min_{\mathbf{v} \in V_0} \|\mathbf{x} - \mathbf{v}\|.$$

That is, the distance from $\mathbf{x}$ to the subspace V_0 is $\|\mathbf{o}\|$.


x
o
p
V_0

Let V be an inner product space. Let $\mathbf{p}$ be the orthogonal projection of a vector $\mathbf{x} \in V$ onto a finite-dimensional subspace V_0.

If V_0 is a one-dimensional subspace spanned by a vector $\mathbf{v}$ then $\mathbf{p} = \dfrac{\langle \mathbf{x}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}\mathbf{v}$.

If $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ is an orthogonal basis for V_0 then

$$\mathbf{p} = \frac{\langle \mathbf{x}, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 + \frac{\langle \mathbf{x}, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle}\mathbf{v}_2 + \cdots + \frac{\langle \mathbf{x}, \mathbf{v}_n \rangle}{\langle \mathbf{v}_n, \mathbf{v}_n \rangle}\mathbf{v}_n.$$

Indeed, $\langle \mathbf{p}, \mathbf{v}_i \rangle = \displaystyle\sum_{j=1}^{n} \frac{\langle \mathbf{x}, \mathbf{v}_j \rangle}{\langle \mathbf{v}_j, \mathbf{v}_j \rangle}\langle \mathbf{v}_j, \mathbf{v}_i \rangle = \frac{\langle \mathbf{x}, \mathbf{v}_i \rangle}{\langle \mathbf{v}_i, \mathbf{v}_i \rangle}\langle \mathbf{v}_i, \mathbf{v}_i \rangle = \langle \mathbf{x}, \mathbf{v}_i \rangle$

$\implies \langle \mathbf{x}-\mathbf{p}, \mathbf{v}_i \rangle = 0 \implies \mathbf{x}-\mathbf{p} \perp \mathbf{v}_i \implies \mathbf{x}-\mathbf{p} \perp V_0.$

The Gram-Schmidt orthogonalization process

Let V be a vector space with an inner product. Suppose $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ is a basis for V. Let

$$\mathbf{v}_1 = \mathbf{x}_1,$$

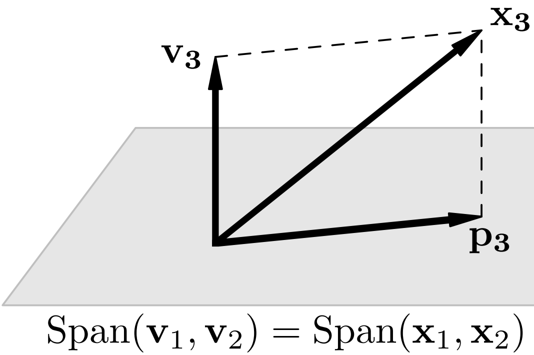
$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1,$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\langle \mathbf{x}_3, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1 - \frac{\langle \mathbf{x}_3, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle} \mathbf{v}_2,$$

....

$$\mathbf{v}_n = \mathbf{x}_n - \frac{\langle \mathbf{x}_n, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1 - \cdots - \frac{\langle \mathbf{x}_n, \mathbf{v}_{n-1} \rangle}{\langle \mathbf{v}_{n-1}, \mathbf{v}_{n-1} \rangle} \mathbf{v}_{n-1}.$$

Then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is an orthogonal basis for V.


$\mathbf{x_3}$
$\mathbf{v_3}$
$\mathbf{p_3}$
$\mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2) = \mathrm{Span}(\mathbf{x}_1, \mathbf{x}_2)$

Any basis	$\longrightarrow$	Orthogonal basis
$\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$		$\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$

Properties of the Gram-Schmidt process:

- $\mathbf{v}_k = \mathbf{x}_k - (\alpha_1 \mathbf{x}_1 + \cdots + \alpha_{k-1} \mathbf{x}_{k-1})$, $1 \leq k \leq n$;
- the span of $\mathbf{v}_1, \dots, \mathbf{v}_k$ is the same as the span of $\mathbf{x}_1, \dots, \mathbf{x}_k$;
- $\mathbf{v}_k$ is orthogonal to $\mathbf{x}_1, \dots, \mathbf{x}_{k-1}$;
- $\mathbf{v}_k = \mathbf{x}_k - \mathbf{p}_k$, where $\mathbf{p}_k$ is the orthogonal projection of the vector $\mathbf{x}_k$ on the subspace spanned by $\mathbf{x}_1, \dots, \mathbf{x}_{k-1}$;
- $\|\mathbf{v}_k\|$ is the distance from $\mathbf{x}_k$ to the subspace spanned by $\mathbf{x}_1, \dots, \mathbf{x}_{k-1}$.

Normalization

Let V be a vector space with an inner product.
Suppose $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is an orthogonal basis for V.

Let $\mathbf{w}_1 = \dfrac{\mathbf{v}_1}{\|\mathbf{v}_1\|}$, $\mathbf{w}_2 = \dfrac{\mathbf{v}_2}{\|\mathbf{v}_2\|}$,..., $\mathbf{w}_n = \dfrac{\mathbf{v}_n}{\|\mathbf{v}_n\|}$.

Then $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n$ is an orthonormal basis for V.

Theorem Any finite-dimensional vector space with an inner product has an orthonormal basis.

Remark. An infinite-dimensional vector space with an inner product may or may not have an orthonormal basis.

Orthogonalization / Normalization

An alternative form of the Gram-Schmidt process combines orthogonalization with normalization.

Suppose $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ is a basis for an inner product space V. Let

$$\mathbf{v}_1 = \mathbf{x}_1, \quad \mathbf{w}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|},$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \langle \mathbf{x}_2, \mathbf{w}_1 \rangle \mathbf{w}_1, \quad \mathbf{w}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|},$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \langle \mathbf{x}_3, \mathbf{w}_1 \rangle \mathbf{w}_1 - \langle \mathbf{x}_3, \mathbf{w}_2 \rangle \mathbf{w}_2, \quad \mathbf{w}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\mathbf{v}_n = \mathbf{x}_n - \langle \mathbf{x}_n, \mathbf{w}_1 \rangle \mathbf{w}_1 - \cdots - \langle \mathbf{x}_n, \mathbf{w}_{n-1} \rangle \mathbf{w}_{n-1},$$
$$\mathbf{w}_n = \frac{\mathbf{v}_n}{\|\mathbf{v}_n\|}.$$

Then $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n$ is an orthonormal basis for V.