MATH 304

Linear Algebra

Lecture 36: Complexification. Symmetric and orthogonal matrices.

Complex numbers

$\mathbb{C}$: complex numbers.

Complex number: $\boxed{z = x + iy,}$

where $x, y \in \mathbb{R}$ and $i^2 = -1$.

$i = \sqrt{-1}$: imaginary unit

Alternative notation: $z = x + yi$.

x = real part of z,
iy = imaginary part of z

$y = 0 \implies z = x$ (real number)
$x = 0 \implies z = iy$ (purely imaginary number)

We add, subtract, and multiply complex numbers as polynomials in i (but keep in mind that $i^2 = -1$).

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2),$$

$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2),$$

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1).$$

Given $z = x + iy$, the **complex conjugate** of z is $\bar{z} = x - iy$. The **modulus** of z is $|z| = \sqrt{x^2 + y^2}$.

$$z\bar{z} = (x + iy)(x - iy) = x^2 - (iy)^2 = x^2 + y^2 = |z|^2.$$

$$z^{-1} = \frac{\bar{z}}{|z|^2}, \qquad (x + iy)^{-1} = \frac{x - iy}{x^2 + y^2}.$$

Complex exponentials

Definition. For any $z \in \mathbb{C}$ let
$$e^z = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots$$

Remark. A sequence of complex numbers $z_1 = x_1 + iy_1,\ z_2 = x_2 + iy_2, \ldots$ converges to $z = x + iy$ if $x_n \to x$ and $y_n \to y$ as $n \to \infty$.

Theorem 1 If $z = x + iy,\ x, y \in \mathbb{R},$ then
$$e^z = e^x(\cos y + i \sin y).$$

In particular, $e^{i\phi} = \cos\phi + i\sin\phi,\ \phi \in \mathbb{R}.$

Theorem 2 $e^{z+w} = e^z \cdot e^w$ for all $z, w \in \mathbb{C}$.

Proposition $e^{i\phi} = \cos\phi + i\sin\phi$ for all $\phi \in \mathbb{R}$.

Proof: $e^{i\phi} = 1 + i\phi + \frac{(i\phi)^2}{2!} + \cdots + \frac{(i\phi)^n}{n!} + \cdots$

The sequence $1, i, i^2, i^3, \ldots, i^n, \ldots$ is periodic:
$\underbrace{1, i, -1, -i}, \underbrace{1, i, -1, -i}, \ldots$

It follows that

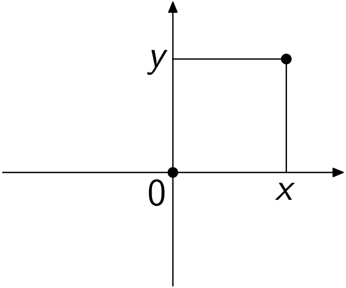
$$e^{i\phi} = 1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} - \cdots + (-1)^k \frac{\phi^{2k}}{(2k)!} + \cdots$$

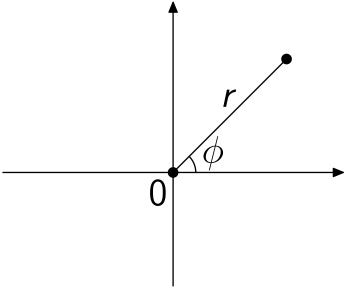
$$+\ i\left(\phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - \cdots + (-1)^k \frac{\phi^{2k+1}}{(2k+1)!} + \cdots\right)$$

$$= \cos\phi + i\sin\phi.$$

Geometric representation

Any complex number $z = x + iy$ is represented by the vector/point $(x, y) \in \mathbb{R}^2$.





$$x = r\cos\phi,\ \ y = r\sin\phi \implies z = r(\cos\phi + i\sin\phi) = re^{i\phi}$$

If $z_1 = r_1 e^{i\phi_1}$ and $z_2 = r_2 e^{i\phi_2}$, then
$z_1 z_2 = r_1 r_2 e^{i(\phi_1+\phi_2)}$, $z_1/z_2 = (r_1/r_2)e^{i(\phi_1-\phi_2)}$.

Fundamental Theorem of Algebra
Any polynomial of degree $n \geq 1$, with complex coefficients, has exactly n roots (counting with multiplicities).

Equivalently, if

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

where $a_i \in \mathbb{C}$ and $a_n \neq 0$, then there exist complex numbers $z_1, z_2, \ldots, z_n$ such that

$$p(z) = a_n (z - z_1)(z - z_2) \ldots (z - z_n).$$

Complex eigenvalues/eigenvectors

Example. $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. $\det(A - \lambda I) = \lambda^2 + 1$.

Characteristic roots: $\lambda_1 = i$ and $\lambda_2 = -i$.

Associated eigenvectors: $\mathbf{v}_1 = (1, -i)$ and $\mathbf{v}_2 = (1, i)$.

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} i \\ 1 \end{pmatrix} = i \begin{pmatrix} 1 \\ -i \end{pmatrix},$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} -i \\ 1 \end{pmatrix} = -i \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

$\mathbf{v}_1$, $\mathbf{v}_2$ is a basis of eigenvectors. *In which space?*

Complexification

Instead of the real vector space $\mathbb{R}^2$, we consider a *complex vector space* $\mathbb{C}^2$ (all complex numbers are admissible as scalars).

The linear operator $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(\mathbf{x}) = A\mathbf{x}$ is extended to a *complex linear operator* $F : \mathbb{C}^2 \to \mathbb{C}^2$, $F(\mathbf{x}) = A\mathbf{x}$.

The vectors $\mathbf{v}_1 = (1, -i)$ and $\mathbf{v}_2 = (1, i)$ form a basis for $\mathbb{C}^2$.

$\mathbb{C}^2$ is also a real vector space (of real dimension 4). The standard real basis for $\mathbb{C}^2$ is $\mathbf{e}_1 = (1, 0)$, $\mathbf{e}_2 = (0, 1)$, $i\mathbf{e}_1 = (i, 0)$, $i\mathbf{e}_2 = (0, i)$. The matrix of the operator F with respect to this basis has a block structure $\begin{pmatrix} A & O \\ O & A \end{pmatrix}$.

Dot product of complex vectors

Dot product of real vectors
$\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$:

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

Dot product of complex vectors
$\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{C}^n$:

$$\mathbf{x} \cdot \mathbf{y} = x_1 \overline{y_1} + x_2 \overline{y_2} + \cdots + x_n \overline{y_n}.$$

If $z = r + it$ $(t, s \in \mathbb{R})$ then $\overline{z} = r - it$,
$z\overline{z} = r^2 + t^2 = |z|^2$.

Hence $\mathbf{x} \cdot \mathbf{x} = |x_1|^2 + |x_2|^2 + \cdots + |x_n|^2 \geq 0$.

Also, $\mathbf{x} \cdot \mathbf{x} = 0$ if and only if $\mathbf{x} = \mathbf{0}$.

The norm is defined by $\|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$.

Normal matrices

Definition. An $n \times n$ matrix A is called

- **symmetric** if $A^T = A$;
- **orthogonal** if $AA^T = A^T A = I$, i.e., $A^T = A^{-1}$;
- **normal** if $AA^T = A^T A$.

Theorem Let A be an $n \times n$ matrix with real entries. Then
(a) A is normal $\iff$ there exists an orthonormal basis for $\mathbb{C}^n$ consisting of eigenvectors of A;
(b) A is symmetric $\iff$ there exists an orthonormal basis for $\mathbb{R}^n$ consisting of eigenvectors of A.

Example. $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

- A is symmetric.
- A has three eigenvalues: 0, 2, and 3.
- Associated eigenvectors are $\mathbf{v}_1 = (-1, 0, 1)$, $\mathbf{v}_2 = (1, 0, 1)$, and $\mathbf{v}_3 = (0, 1, 0)$, respectively.
- Vectors $\frac{1}{\sqrt{2}}\mathbf{v}_1, \frac{1}{\sqrt{2}}\mathbf{v}_2, \mathbf{v}_3$ form an orthonormal basis for $\mathbb{R}^3$.

Theorem Suppose A is a normal matrix. Then for any $\mathbf{x} \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$ one has

$$A\mathbf{x} = \lambda\mathbf{x} \iff A^T\mathbf{x} = \overline{\lambda}\mathbf{x}.$$

Thus any normal matrix A shares with A^T all real eigenvalues and the corresponding eigenvectors. Also, $A\mathbf{x} = \lambda\mathbf{x} \iff A\overline{\mathbf{x}} = \overline{\lambda}\,\overline{\mathbf{x}}$ for any matrix A with real entries.

Corollary All eigenvalues λ of a symmetric matrix are real ($\overline{\lambda} = \lambda$). All eigenvalues λ of an orthogonal matrix satisfy $\overline{\lambda} = \lambda^{-1} \iff |\lambda| = 1$.

Why are orthogonal matrices called so?

Theorem Given an $n \times n$ matrix A, the following conditions are equivalent:
(i) A is orthogonal: $A^T = A^{-1}$;
(ii) columns of A form an orthonormal basis for $\mathbb{R}^n$;
(iii) rows of A form an orthonormal basis for $\mathbb{R}^n$.

Proof: Entries of the matrix $A^T A$ are dot products of columns of A. Entries of AA^T are dot products of rows of A.

Thus an orthogonal matrix is the transition matrix from one orthonormal basis to another.

Example. $A_\phi = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$.

- $A_\phi A_\psi = A_{\phi+\psi}$
- $A_\phi^{-1} = A_{-\phi} = A_\phi^T$
- A_ϕ is orthogonal
- $\det(A_\phi - \lambda I) = (\cos\phi - \lambda)^2 + \sin^2\phi$.
- Eigenvalues: $\lambda_1 = \cos\phi + i\sin\phi = e^{i\phi}$, $\lambda_2 = \cos\phi - i\sin\phi = e^{-i\phi}$.
- Associated eigenvectors: $\mathbf{v}_1 = (1, -i)$, $\mathbf{v}_2 = (1, i)$.
- Vectors $\frac{1}{\sqrt{2}}\mathbf{v}_1$ and $\frac{1}{\sqrt{2}}\mathbf{v}_2$ form an orthonormal basis for $\mathbb{C}^2$.

Consider a linear operator $L : \mathbb{R}^n \to \mathbb{R}^n$, $L(\mathbf{x}) = A\mathbf{x}$, where A is an $n \times n$ orthogonal matrix.

Theorem There exists an orthonormal basis for $\mathbb{R}^n$ such that the matrix of L relative to this basis has a diagonal block structure

$$\begin{pmatrix} D_{\pm 1} & O & \dots & O \\ O & R_1 & \dots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \dots & R_k \end{pmatrix},$$

where $D_{\pm 1}$ is a diagonal matrix whose diagonal entries are equal to 1 or -1, and

$$R_j = \begin{pmatrix} \cos\phi_j & -\sin\phi_j \\ \sin\phi_j & \cos\phi_j \end{pmatrix}, \;\; \phi_j \in \mathbb{R}.$$