

MATH 304
Linear Algebra

Lecture 38:
Orthogonal polynomials.

Problem. Approximate the function $f(x) = e^x$ on the interval $[-1, 1]$ by a quadratic polynomial.

The best approximation would be a polynomial $p(x)$ that minimizes the distance relative to the uniform norm:

$$\|f - p\|_{\infty} = \max_{|x| \leq 1} |f(x) - p(x)|.$$

However there is no analytic way to find such a polynomial. Another approach is to find a “least squares” approximation that minimizes the integral norm

$$\|f - p\|_2 = \left(\int_{-1}^1 |f(x) - p(x)|^2 dx \right)^{1/2}.$$

The norm $\| \cdot \|_2$ is induced by the inner product

$$\langle g, h \rangle = \int_{-1}^1 g(x)h(x) dx.$$

Therefore $\|f - p\|_2$ is minimal if p is the orthogonal projection of the function f on the subspace \mathcal{P}_3 of quadratic polynomials.

Suppose that p_0, p_1, p_2 is an orthogonal basis for \mathcal{P}_3 . Then

$$p(x) = \frac{\langle f, p_0 \rangle}{\langle p_0, p_0 \rangle} p_0(x) + \frac{\langle f, p_1 \rangle}{\langle p_1, p_1 \rangle} p_1(x) + \frac{\langle f, p_2 \rangle}{\langle p_2, p_2 \rangle} p_2(x).$$

Orthogonal polynomials

\mathcal{P} : the vector space of all polynomials with real coefficients: $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$.

Basis for \mathcal{P} : $1, x, x^2, \dots, x^n, \dots$

Suppose that \mathcal{P} is endowed with an inner product.

Definition. **Orthogonal polynomials** (relative to the inner product) are polynomials p_0, p_1, p_2, \dots such that $\deg p_n = n$ (p_0 is a nonzero constant) and $\langle p_n, p_m \rangle = 0$ for $n \neq m$.

Orthogonal polynomials can be obtained by applying the *Gram-Schmidt orthogonalization process* to the basis $1, x, x^2, \dots$:

$$p_0(x) = 1,$$

$$p_1(x) = x - \frac{\langle x, p_0 \rangle}{\langle p_0, p_0 \rangle} p_0(x),$$

$$p_2(x) = x^2 - \frac{\langle x^2, p_0 \rangle}{\langle p_0, p_0 \rangle} p_0(x) - \frac{\langle x^2, p_1 \rangle}{\langle p_1, p_1 \rangle} p_1(x),$$

.....

$$p_n(x) = x^n - \frac{\langle x^n, p_0 \rangle}{\langle p_0, p_0 \rangle} p_0(x) - \dots - \frac{\langle x^n, p_{n-1} \rangle}{\langle p_{n-1}, p_{n-1} \rangle} p_{n-1}(x),$$

.....

Then p_0, p_1, p_2, \dots are orthogonal polynomials.

Theorem (a) Orthogonal polynomials always exist.

(b) The orthogonal polynomial of a fixed degree is unique up to scaling.

(c) A polynomial $p \neq 0$ is an orthogonal polynomial if and only if $\langle p, q \rangle = 0$ for any polynomial q with $\deg q < \deg p$.

(d) A polynomial $p \neq 0$ is an orthogonal polynomial if and only if $\langle p, x^k \rangle = 0$ for any $0 \leq k < \deg p$.

Proof of statement (b): Suppose that P and R are two orthogonal polynomials of the same degree n . Then

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \text{ and}$$

$$R(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0, \text{ where } a_n, b_n \neq 0.$$

Consider a polynomial $Q(x) = b_n P(x) - a_n R(x)$. By construction, $\deg Q < n$. It follows from statement (c) that $\langle P, Q \rangle = \langle R, Q \rangle = 0$. Then

$$\langle Q, Q \rangle = \langle b_n P - a_n R, Q \rangle = b_n \langle P, Q \rangle - a_n \langle R, Q \rangle = 0,$$

which means that $Q = 0$. Thus $R(x) = (a_n^{-1} b_n) P(x)$.

Example. $\langle p, q \rangle = \int_{-1}^1 p(x)q(x) dx.$

Note that $\langle x^n, x^m \rangle = 0$ if $m + n$ is odd.

Hence $p_{2k}(x)$ contains only even powers of x while $p_{2k+1}(x)$ contains only odd powers of x .

$$p_0(x) = 1,$$

$$p_1(x) = x,$$

$$p_2(x) = x^2 - \frac{\langle x^2, 1 \rangle}{\langle 1, 1 \rangle} = x^2 - \frac{1}{3},$$

$$p_3(x) = x^3 - \frac{\langle x^3, x \rangle}{\langle x, x \rangle} x = x^3 - \frac{3}{5}x.$$

p_0, p_1, p_2, \dots are called the **Legendre polynomials**.

Instead of normalization, the orthogonal polynomials are subject to **standardization**.

The standardization for the Legendre polynomials is $P_n(1) = 1$. In particular, $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2 - 1)$, $P_3(x) = \frac{1}{2}(5x^3 - 3x)$.

Problem. Find $P_4(x)$.

Let $P_4(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$.

We know that $P_4(1) = 1$ and $\langle P_4, x^k \rangle = 0$ for $0 \leq k \leq 3$.

$$P_4(1) = a_4 + a_3 + a_2 + a_1 + a_0,$$

$$\langle P_4, 1 \rangle = \frac{2}{5}a_4 + \frac{2}{3}a_2 + 2a_0, \quad \langle P_4, x \rangle = \frac{2}{5}a_3 + \frac{2}{3}a_1,$$

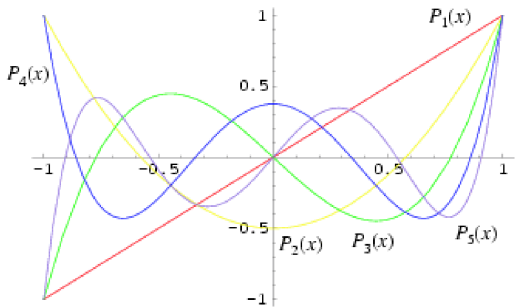
$$\langle P_4, x^2 \rangle = \frac{2}{7}a_4 + \frac{2}{5}a_2 + \frac{2}{3}a_0, \quad \langle P_4, x^3 \rangle = \frac{2}{7}a_3 + \frac{2}{5}a_1.$$

$$\begin{cases} a_4 + a_3 + a_2 + a_1 + a_0 = 1 \\ \frac{2}{5}a_4 + \frac{2}{3}a_2 + 2a_0 = 0 \\ \frac{2}{5}a_3 + \frac{2}{3}a_1 = 0 \\ \frac{2}{7}a_4 + \frac{2}{5}a_2 + \frac{2}{3}a_0 = 0 \\ \frac{2}{7}a_3 + \frac{2}{5}a_1 = 0 \end{cases}$$

$$\begin{cases} \frac{2}{5}a_3 + \frac{2}{3}a_1 = 0 \\ \frac{2}{7}a_3 + \frac{2}{5}a_1 = 0 \end{cases} \implies a_1 = a_3 = 0$$

$$\begin{cases} a_4 + a_2 + a_0 = 1 \\ \frac{2}{5}a_4 + \frac{2}{3}a_2 + 2a_0 = 0 \\ \frac{2}{7}a_4 + \frac{2}{5}a_2 + \frac{2}{3}a_0 = 0 \end{cases} \iff \begin{cases} a_4 = \frac{35}{8} \\ a_2 = -\frac{30}{8} \\ a_0 = \frac{3}{8} \end{cases}$$

Thus $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$.



Legendre polynomials

Problem. Find a quadratic polynomial that is the best least squares fit to the function $f(x) = |x|$ on the interval $[-1, 1]$.

The best least squares fit is a polynomial $p(x)$ that minimizes the distance relative to the integral norm

$$\|f - p\| = \left(\int_{-1}^1 |f(x) - p(x)|^2 dx \right)^{1/2}$$

over all polynomials of degree 2.

The norm $\|f - p\|$ is minimal if p is the orthogonal projection of the function f on the subspace \mathcal{P}_3 of polynomials of degree at most 2.

The Legendre polynomials P_0, P_1, P_2 form an orthogonal basis for \mathcal{P}_3 . Therefore

$$p(x) = \frac{\langle f, P_0 \rangle}{\langle P_0, P_0 \rangle} P_0(x) + \frac{\langle f, P_1 \rangle}{\langle P_1, P_1 \rangle} P_1(x) + \frac{\langle f, P_2 \rangle}{\langle P_2, P_2 \rangle} P_2(x).$$

$$\langle f, P_0 \rangle = \int_{-1}^1 |x| dx = 2 \int_0^1 x dx = 1,$$

$$\langle f, P_1 \rangle = \int_{-1}^1 |x| x dx = 0,$$

$$\langle f, P_2 \rangle = \int_{-1}^1 |x| \frac{3x^2 - 1}{2} dx = \int_0^1 x(3x^2 - 1) dx = \frac{1}{4},$$

$$\langle P_0, P_0 \rangle = \int_{-1}^1 dx = 2, \quad \langle P_2, P_2 \rangle = \int_{-1}^1 \left(\frac{3x^2 - 1}{2} \right)^2 dx = \frac{2}{5}.$$

$$\text{In general, } \langle P_n, P_n \rangle = \frac{2}{2n+1}.$$

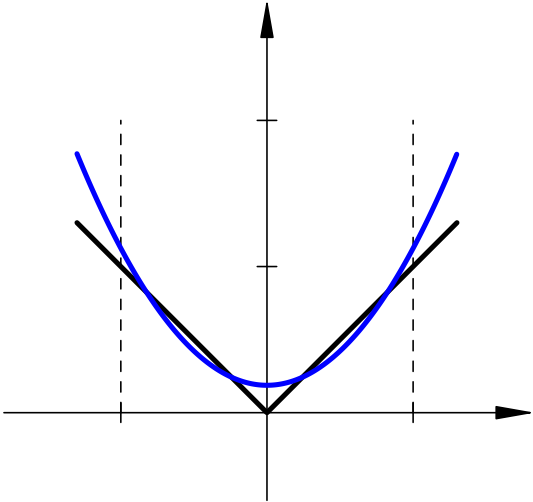
Problem. Find a quadratic polynomial that is the best least squares fit to the function $f(x) = |x|$ on the interval $[-1, 1]$.

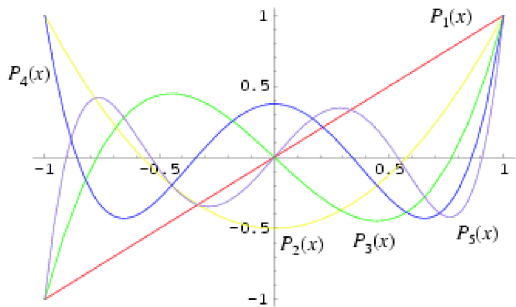
Solution:
$$p(x) = \frac{1}{2}P_0(x) + \frac{5}{8}P_2(x)$$
$$= \frac{1}{2} + \frac{5}{16}(3x^2 - 1) = \frac{3}{16}(5x^2 + 1).$$

Recurrent formula for the Legendre polynomials:

$$(n + 1)P_{n+1}(x) = (2n + 1)xP_n(x) - nP_{n-1}(x).$$

For example, $4P_4(x) = 7xP_3(x) - 3P_2(x)$.





Legendre polynomials

Definition. **Chebyshev polynomials** T_0, T_1, T_2, \dots are orthogonal polynomials relative to the inner product

$$\langle p, q \rangle = \int_{-1}^1 \frac{p(x)q(x)}{\sqrt{1-x^2}} dx,$$

with the standardization $T_n(1) = 1$.

Remark. “T” is like in “Tschebyscheff”.

Change of variable in the integral: $x = \cos \phi$.

$$\begin{aligned} \langle p, q \rangle &= - \int_0^\pi \frac{p(\cos \phi) q(\cos \phi)}{\sqrt{1 - \cos^2 \phi}} \cos' \phi d\phi \\ &= \int_0^\pi p(\cos \phi) q(\cos \phi) d\phi. \end{aligned}$$

Theorem. $T_n(\cos \phi) = \cos n\phi$.

$$\begin{aligned}\langle T_n, T_m \rangle &= \int_0^\pi T_n(\cos \phi) T_m(\cos \phi) d\phi \\ &= \int_0^\pi \cos(n\phi) \cos(m\phi) d\phi = 0 \quad \text{if } n \neq m.\end{aligned}$$

Recurrent formula: $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$.

$$T_0(x) = 1, \quad T_1(x) = x,$$

$$T_2(x) = 2x^2 - 1,$$

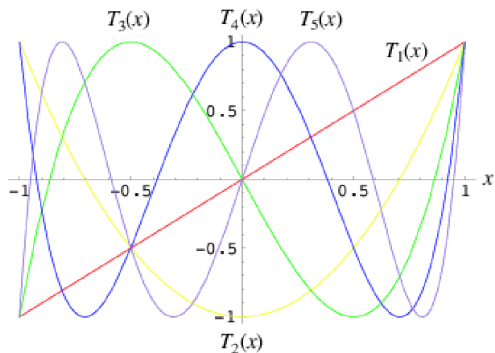
$$T_3(x) = 4x^3 - 3x,$$

$$T_4(x) = 8x^4 - 8x^2 + 1, \dots$$

That is, $\cos 2\phi = 2 \cos^2 \phi - 1$,

$\cos 3\phi = 4 \cos^3 \phi - 3 \cos \phi$,

$\cos 4\phi = 8 \cos^4 \phi - 8 \cos^2 \phi + 1, \dots$



Chebyshev polynomials