# Sample problems for Test 1: Solutions

**Any problem may be altered or replaced by a different one!**

**Problem 1 (15 pts.)** Find a quadratic polynomial $p(x)$ such that $p(1) = 1$, $p(2) = 3$, and $p(3) = 7$.

Let $p(x) = ax^2 + bx + c$. Then $p(1) = a + b + c$, $p(2) = 4a + 2b + c$, and $p(3) = 9a + 3b + c$. The coefficients $a$, $b$, and $c$ have to be chosen so that

$$\begin{cases} a + b + c = 1, \\ 4a + 2b + c = 3, \\ 9a + 3b + c = 7. \end{cases}$$

We solve this system of linear equations using elementary operations:

$$\begin{cases} a+b+c=1 \\ 4a+2b+c=3 \\ 9a+3b+c=7 \end{cases} \iff \begin{cases} a+b+c=1 \\ 3a+b=2 \\ 9a+3b+c=7 \end{cases} \iff \begin{cases} a+b+c=1 \\ 3a+b=2 \\ 8a+2b=6 \end{cases} \iff \begin{cases} a+b+c=1 \\ 3a+b=2 \\ 4a+b=3 \end{cases}$$

$$\iff \begin{cases} a+b+c=1 \\ 3a+b=2 \\ a=1 \end{cases} \iff \begin{cases} a+b+c=1 \\ b=-1 \\ a=1 \end{cases} \iff \begin{cases} c=1 \\ b=-1 \\ a=1 \end{cases}$$

Thus the desired polynomial is $p(x) = x^2 - x + 1$.

**Problem 2 (25 pts.)** Let $A = \begin{pmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 2 & 0 & -1 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix}$.

(i) Evaluate the determinant of the matrix $A$.

First let us subtract 2 times the fourth column of $A$ from the first column:

$$\begin{vmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 2 & 0 & -1 & 1 \\ 2 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} -1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix}.$$

Now the determinant can be easily expanded by the fourth row:

$$\begin{vmatrix} -1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} -1 & -2 & 4 \\ 2 & 3 & 2 \\ 0 & 0 & -1 \end{vmatrix}.$$

The $3 \times 3$ determinant is easily expanded by the third row:

$$\begin{vmatrix} -1 & -2 & 4 \\ 2 & 3 & 2 \\ 0 & 0 & -1 \end{vmatrix} = (-1) \begin{vmatrix} -1 & -2 \\ 2 & 3 \end{vmatrix}.$$

Thus

$$\det A = -\begin{vmatrix} -1 & -2 \\ 2 & 3 \end{vmatrix} = -1.$$

Another way to evaluate $\det A$ is to reduce the matrix $A$ to the identity matrix using elementary row operations (see below). This requires much more work but we are going to do it anyway, to find the inverse of $A$.

(ii) Find the inverse matrix $A^{-1}$.

First we merge the matrix $A$ with the identity matrix into one $4 \times 8$ matrix

$$(A \,|\, I) = \left(\begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 2 & 3 & 2 & 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right).$$

Then we apply elementary row operations to this matrix until the left part becomes the identity matrix.

Subtract 2 times the first row from the second row:

$$\left(\begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 2 & 3 & 2 & 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 7 & -6 & -2 & -2 & 1 & 0 & 0 \\ 2 & 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right).$$

Subtract 2 times the first row from the third row:

$$\left(\begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 7 & -6 & -2 & -2 & 1 & 0 & 0 \\ 2 & 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 7 & -6 & -2 & -2 & 1 & 0 & 0 \\ 0 & 4 & -9 & -1 & -2 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right).$$

Subtract 2 times the first row from the fourth row:

$$\left(\begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 7 & -6 & -2 & -2 & 1 & 0 & 0 \\ 0 & 4 & -9 & -1 & -2 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 7 & -6 & -2 & -2 & 1 & 0 & 0 \\ 0 & 4 & -9 & -1 & -2 & 0 & 1 & 0 \\ 0 & 4 & -8 & -1 & -2 & 0 & 0 & 1 \end{array}\right).$$

Subtract 2 times the fourth row from the second row:

$$\left(\begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 7 & -6 & -2 & -2 & 1 & 0 & 0 \\ 0 & 4 & -9 & -1 & -2 & 0 & 1 & 0 \\ 0 & 4 & -8 & -1 & -2 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 10 & 0 & 2 & 1 & 0 & -2 \\ 0 & 4 & -9 & -1 & -2 & 0 & 1 & 0 \\ 0 & 4 & -8 & -1 & -2 & 0 & 0 & 1 \end{array}\right).$$

Subtract the fourth row from the third row:

$$\left(\begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 10 & 0 & 2 & 1 & 0 & -2 \\ 0 & 4 & -9 & -1 & -2 & 0 & 1 & 0 \\ 0 & 4 & -8 & -1 & -2 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 10 & 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 4 & -8 & -1 & -2 & 0 & 0 & 1 \end{array}\right).$$

Add 4 times the second row to the fourth row:

$$\left(\begin{array}{rrrr|rrrr} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 10 & 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 4 & -8 & -1 & -2 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{rrrr|rrrr} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 10 & 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 32 & -1 & 6 & 4 & 0 & -7 \end{array}\right).$$

Add 32 times the third row to the fourth row:

$$\left(\begin{array}{rrrr|rrrr} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 10 & 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 32 & -1 & 6 & 4 & 0 & -7 \end{array}\right) \to \left(\begin{array}{rrrr|rrrr} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 10 & 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 6 & 4 & 32 & -39 \end{array}\right).$$

Add 10 times the third row to the second row:

$$\left(\begin{array}{rrrr|rrrr} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 10 & 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 6 & 4 & 32 & -39 \end{array}\right) \to \left(\begin{array}{rrrr|rrrr} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 2 & 1 & 10 & -12 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 6 & 4 & 32 & -39 \end{array}\right).$$

Add the fourth row to the first row:

$$\left(\begin{array}{rrrr|rrrr} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 2 & 1 & 10 & -12 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 6 & 4 & 32 & -39 \end{array}\right) \to \left(\begin{array}{rrrr|rrrr} 1 & -2 & 4 & 0 & 7 & 4 & 32 & -39 \\ 0 & -1 & 0 & 0 & 2 & 1 & 10 & -12 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 6 & 4 & 32 & -39 \end{array}\right).$$

Add 4 times the third row to the first row:

$$\left(\begin{array}{rrrr|rrrr} 1 & -2 & 4 & 0 & 7 & 4 & 32 & -39 \\ 0 & -1 & 0 & 0 & 2 & 1 & 10 & -12 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 6 & 4 & 32 & -39 \end{array}\right) \to \left(\begin{array}{rrrr|rrrr} 1 & -2 & 0 & 0 & 7 & 4 & 36 & -43 \\ 0 & -1 & 0 & 0 & 2 & 1 & 10 & -12 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 6 & 4 & 32 & -39 \end{array}\right).$$

Subtract 2 times the second row from the first row:

$$\left(\begin{array}{rrrr|rrrr} 1 & -2 & 0 & 0 & 7 & 4 & 36 & -43 \\ 0 & -1 & 0 & 0 & 2 & 1 & 10 & -12 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 6 & 4 & 32 & -39 \end{array}\right) \to \left(\begin{array}{rrrr|rrrr} 1 & 0 & 0 & 0 & 3 & 2 & 16 & -19 \\ 0 & -1 & 0 & 0 & 2 & 1 & 10 & -12 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 6 & 4 & 32 & -39 \end{array}\right).$$

Multiply the second, the third, and the fourth rows by $-1$:

$$\left(\begin{array}{rrrr|rrrr} 1 & 0 & 0 & 0 & 3 & 2 & 16 & -19 \\ 0 & -1 & 0 & 0 & 2 & 1 & 10 & -12 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 6 & 4 & 32 & -39 \end{array}\right) \to \left(\begin{array}{rrrr|rrrr} 1 & 0 & 0 & 0 & 3 & 2 & 16 & -19 \\ 0 & 1 & 0 & 0 & -2 & -1 & -10 & 12 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & -6 & -4 & -32 & 39 \end{array}\right).$$

Finally the left part of our $4 \times 8$ matrix is transformed into the identity matrix. Therefore the current right part is the inverse matrix of $A$. Thus

$$A^{-1} = \begin{pmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 2 & 0 & -1 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & 2 & 16 & -19 \\ -2 & -1 & -10 & 12 \\ 0 & 0 & -1 & 1 \\ -6 & -4 & -32 & 39 \end{pmatrix}.$$

As a byproduct, we can evaluate the determinant of $A$. We have transformed $A$ into the identity matrix using elementary row operations. These included no row exchanges and three row multiplications, each time by $-1$. It follows that $\det I = (-1)^3 \det A$. Hence $\det A = -\det I = -1$.

**Problem 3 (20 pts.)** Determine which of the following subsets of $\mathbb{R}^3$ are subspaces. Briefly explain.

(i) The set $S_1$ of vectors $(x, y, z) \in \mathbb{R}^3$ such that $xyz = 0$.
(ii) The set $S_2$ of vectors $(x, y, z) \in \mathbb{R}^3$ such that $x + y + z = 0$.
(iii) The set $S_3$ of vectors $(x, y, z) \in \mathbb{R}^3$ such that $y^2 + z^2 = 0$.
(iv) The set $S_4$ of vectors $(x, y, z) \in \mathbb{R}^3$ such that $y^2 - z^2 = 0$.

A subset of $\mathbb{R}^3$ is a subspace if it is closed under addition and scalar multiplication. Besides, a subspace must not be empty.

It is easy to see that each of the sets $S_1$, $S_2$, $S_3$, and $S_4$ contains the zero vector $(0, 0, 0)$ and all these sets are closed under scalar multiplication.

The set $S_1$ is the union of three planes $x = 0$, $y = 0$, and $z = 0$. It is not closed under addition as the following example shows: $(1, 1, 0) + (0, 0, 1) = (1, 1, 1)$.

$S_2$ is a plane passing through the origin. Obviously, it is closed under addition.

The condition $y^2 + z^2 = 0$ is equivalent to $y = z = 0$. Hence $S_3$ is a line passing through the origin. It is closed under addition.

Since $y^2 - z^2 = (y - z)(y + z)$, the set $S_4$ is the union of two planes $y - z = 0$ and $y + z = 0$. The following example shows that $S_4$ is not closed under addition: $(0, 1, 1) + (0, 1, -1) = (0, 2, 0)$.

Thus $S_2$ and $S_3$ are subspaces of $\mathbb{R}^3$ while $S_1$ and $S_4$ are not.

**Problem 4 (30 pts.)** Let $B = \begin{pmatrix} 0 & -1 & 4 & 1 \\ 1 & 1 & 2 & -1 \\ -3 & 0 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix}$.

(i) Find the rank and the nullity of the matrix $B$.

The rank (dimension of the row space) and the nullity (dimension of the nullspace) of a matrix are preserved under elementary row operations. We apply such operations to convert the matrix $B$ into row echelon form.

First interchange the first row with the second row:

$$\begin{pmatrix} 0 & -1 & 4 & 1 \\ 1 & 1 & 2 & -1 \\ -3 & 0 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & -1 & 4 & 1 \\ -3 & 0 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix}.$$

Add 3 times the first row to the third row, then subtract 2 times the first row from the fourth row:

$$\begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & -1 & 4 & 1 \\ -3 & 0 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & -1 & 4 & 1 \\ 0 & 3 & 5 & -3 \\ 2 & -1 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & -1 & 4 & 1 \\ 0 & 3 & 5 & -3 \\ 0 & -3 & -4 & 3 \end{pmatrix}.$$

Multiply the second row by $-1$:

$$\begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & -1 & 4 & 1 \\ 0 & 3 & 5 & -3 \\ 0 & -3 & -4 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 3 & 5 & -3 \\ 0 & -3 & -4 & 3 \end{pmatrix}.$$

Add the fourth row to the third row:

$$\begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 3 & 5 & -3 \\ 0 & -3 & -4 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -3 & -4 & 3 \end{pmatrix}.$$

Add 3 times the second row to the fourth row:

$$\begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -3 & -4 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -16 & 0 \end{pmatrix}.$$

Add 16 times the third row to the fourth row:

$$\begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -16 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now that the matrix is in row echelon form, its rank equals the number of nonzero rows, which is 3. Since

$$(\text{rank of } B) + (\text{nullity of } B) = (\text{the number of columns of } B) = 4,$$

it follows that the nullity of $B$ equals 1.

(ii) Find a basis for the row space of $B$, then extend this basis to a basis for $\mathbb{R}^4$.

The row space of a matrix is invariant under elementary row operations. Therefore the row space of the matrix $B$ is the same as the row space of its row echelon form

$$\begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The nonzero rows of the latter are linearly independent so that they form a basis for its row space. Hence the vectors $\mathbf{v}_1 = (1, 1, 2, -1)$, $\mathbf{v}_2 = (0, 1, -4, -1)$, and $\mathbf{v}_3 = (0, 0, 1, 0)$ form a basis for the row space of $B$.

To extend the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to a basis for $\mathbb{R}^4$, we need a vector $\mathbf{v}_4 \in \mathbb{R}^4$ that is not a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. It is known that at least one of the vectors $\mathbf{e}_1 = (1, 0, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0, 0)$, $\mathbf{e}_3 = (0, 0, 1, 0)$, and $\mathbf{e}_4 = (0, 0, 0, 1)$ can be chosen as $\mathbf{v}_4$. In particular, the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{e}_4$ form a basis for $\mathbb{R}^4$. This follows from the fact that the $4 \times 4$ matrix whose rows are these vectors is not singular:

$$\begin{vmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1 \neq 0.$$

(iii) Find a basis for the nullspace of $B$.

The nullspace of $B$ is the solution set of the system of linear homogeneous equations with $B$ as the coefficient matrix. To solve the system, we convert the matrix $B$ to reduced row echelon form. The row echelon form of $B$ has been obtained earlier:

$$\begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Add 4 times the third row to the second row, then subtract 2 times the third row from the first row:

$$\begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Subtract the second row from the first row:

$$\begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We have obtained the reduced row echelon form of the matrix $B$. Its nullspace is the same as the nullspace of $B$. Hence a vector $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ belongs to the nullspace of $B$ if and only if

$$\begin{cases} x_1 = 0, \\ x_2 - x_4 = 0, \\ x_3 = 0 \end{cases} \iff \begin{cases} x_1 = 0, \\ x_2 = x_4, \\ x_3 = 0. \end{cases}$$

The general solution of this system is $(x_1, x_2, x_3, x_4) = (0, t, 0, t) = t(0, 1, 0, 1)$, $t \in \mathbb{R}$. Thus the nullspace of the matrix $B$ is spanned by the vector $(0, 1, 0, 1)$. This vector forms a basis for the nullspace.

**Bonus Problem 5 (15 pts.)** Show that the functions $f_1(x) = x$, $f_2(x) = xe^x$, and $f_3(x) = e^{-x}$ are linearly independent in the vector space $C^\infty(\mathbb{R})$.

Suppose that $af_1(x) + bf_2(x) + cf_3(x) = 0$ for all $x \in \mathbb{R}$, where $a, b, c$ are constants. We have to show that $a = b = c = 0$.

Differentiating the identity $af_1(x) + bf_2(x) + cf_3(x) = 0$ four times, we obtain four more identities:

$$\begin{aligned} ax + bxe^x + ce^{-x} &= 0, \\ a + be^x + bxe^x - ce^{-x} &= 0, \\ 2be^x + bxe^x + ce^{-x} &= 0, \\ 3be^x + bxe^x - ce^{-x} &= 0, \\ 4be^x + bxe^x + ce^{-x} &= 0. \end{aligned}$$

Subtracting the third identity from the fifth one, we obtain $2be^x = 0$, which implies that $b = 0$. Substituting $b = 0$ in the third identity, we obtain $ce^{-x} = 0$, which implies that $c = 0$. Substituting $b = 0$ and $c = 0$ in the second identity, we obtain $a = 0$.

*Alternative solution:* Suppose that $ax + bxe^x + ce^{-x} = 0$ for all $x \in \mathbb{R}$, where $a, b, c$ are constants. We have to show that $a = b = c = 0$.

For any $x \neq 0$ divide both sides of the identity by $xe^x$:

$$ae^{-x} + b + cx^{-1}e^{-2x} = 0.$$

Note that $e^{-x} \to 0$ and $x^{-1}e^{-2x} \to 0$ as $x \to +\infty$. Hence the left-hand side approaches $b$ as $x \to +\infty$. It follows that $b = 0$. Now $ax + ce^{-x} = 0$ for all $x \in \mathbb{R}$. For any $x \neq 0$ divide both sides of the latter identity by $x$:

$$a + cx^{-1}e^{-x} = 0.$$

Since $x^{-1}e^{-x} \to 0$ as $x \to +\infty$, the left-hand side approaches $a$ as $x \to +\infty$. It follows that $a = 0$. Then $ce^{-x} = 0$, which implies that $c = 0$.

**Bonus Problem 6 (15 pts.)** Let $V$ be a finite-dimensional vector space and $V_0$ be a proper subspace of $V$ (where proper means that $V_0 \neq V$). Prove that $\dim V_0 < \dim V$.

Any linearly independent set in a vector space can be extended to a basis. Since the vector space $V$ is finite dimensional, it does not admit infinitely many linearly independent vectors. Clearly, the same is true for the subspace $V_0$. It follows that $V_0$ is also finite-dimensional.

Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ be a basis for $V_0$. The vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are linearly independent in $V$ since they are linearly independent in $V_0$. Therefore we can extend this collection of vectors to a basis for $V$ by adding some vectors $\mathbf{w}_1, \dots, \mathbf{w}_m$. As $V_0 \neq V$, we do need to add some vectors, i.e., $m \geq 1$. Thus $\dim V_0 = k$ and $\dim V = k + m > k$.