MATH 304

Linear Algebra

Lecture 7: Evaluation of determinants. The Vandermonde determinant. Cramer's rule.

Determinants

Determinant is a scalar assigned to each square matrix.

Notation. The determinant of a matrix $A = (a_{ij})_{1 \le i,j \le n}$ is denoted $\det A$ or

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}.$$

Principal property: $\det A \neq 0$ if and only if a system of linear equations with the coefficient matrix A has a unique solution. Equivalently, $\det A \neq 0$ if and only if the matrix A is invertible.

Explicit definition in low dimensions

Definition. $\det(a) = a, \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc,$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - \\ - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.$$

$$+: \begin{pmatrix} \boxed{*} & * & * \\ * & \boxed{*} & * \\ * & * & \boxed{*} \end{pmatrix}, \begin{pmatrix} * & \boxed{*} & * \\ * & * & \boxed{*} \\ \boxed{*} & * & * \end{pmatrix}, \begin{pmatrix} * & * & \boxed{*} \\ \boxed{*} & * & * \\ * & \boxed{*} & * \end{pmatrix}.$$

$$-: \begin{pmatrix} * & * & \boxed{*} \\ * & \boxed{*} & * \\ \boxed{*} & * & * \end{pmatrix}, \begin{pmatrix} * & \boxed{*} & * \\ \boxed{*} & * & * \\ * & * & \boxed{*} \end{pmatrix}, \begin{pmatrix} \boxed{*} & * & * \\ * & * & \boxed{*} \\ * & \boxed{*} & * \end{pmatrix}.$$

Properties of determinants

Determinants and elementary row operations:

- if a row of a matrix is multiplied by a scalar r, the determinant is also multiplied by r;
- if we add a row of a matrix multiplied by a scalar to another row, the determinant remains the same;
- if we interchange two rows of a matrix, the determinant changes its sign.

Properties of determinants

Tests for singularity:

- if a matrix A has a zero row then $\det A = 0$;
- if a matrix A has two identical rows then $\det A = 0$;
- if a matrix has two proportional rows then $\det A = 0$.

Properties of determinants

Special matrices:

- $\det I = 1$;
- the determinant of a diagonal matrix is equal to the product of its diagonal entries;
- the determinant of an upper triangular matrix is equal to the product of its diagonal entries.

Properties of determinants

Determinant of the transpose:

- If A is a square matrix then $\det A^T = \det A$.

Columns vs. rows:

- if one column of a matrix is multiplied by a scalar, the determinant is multiplied by the same scalar;
- adding a scalar multiple of one column to another does not change the determinant;
- interchanging two columns of a matrix changes the sign of its determinant;
- if a matrix A has a zero column or two proportional columns then $\det A = 0$.

Row and column expansions

Given an $n \times n$ matrix $A = (a_{ij})$, let M_{ij} denote the $(n-1)\times(n-1)$ submatrix obtained by deleting the ith row and the jth column of A.

Theorem For any $1 \leq k, m \leq n$ we have that

$$\det A = \sum_{j=1}^{n} (-1)^{k+j} a_{kj} \det M_{kj},$$

(*expansion by kth row*)

$$\det A = \sum_{i=1}^{n} (-1)^{i+m} a_{im} \det M_{im}.$$

(*expansion by mth column*)

Signs for row/column expansions

$$\begin{pmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Example. $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$.

Expansion by the 1st row:

$$\begin{pmatrix} \boxed{1} & * & * \\ * & 5 & 6 \\ * & 8 & 9 \end{pmatrix} \quad \begin{pmatrix} * & \boxed{2} & * \\ 4 & * & 6 \\ 7 & * & 9 \end{pmatrix} \quad \begin{pmatrix} * & * & \boxed{3} \\ 4 & 5 & * \\ 7 & 8 & * \end{pmatrix}$$

$$\det A = 1 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= (5 \cdot 9 - 6 \cdot 8) - 2(4 \cdot 9 - 6 \cdot 7) + 3(4 \cdot 8 - 5 \cdot 7) = 0.$$

Example. $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$

Expansion by the 2nd column:

$$\begin{pmatrix} * & \boxed{2} & * \\ 4 & * & 6 \\ 7 & * & 9 \end{pmatrix} \quad \begin{pmatrix} 1 & * & 3 \\ * & \boxed{5} & * \\ 7 & * & 9 \end{pmatrix} \quad \begin{pmatrix} 1 & * & 3 \\ 4 & * & 6 \\ * & \boxed{8} & * \end{pmatrix}$$

$$\det A = -2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 5 \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} - 8 \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix}$$

$$= -2(4 \cdot 9 - 6 \cdot 7) + 5(1 \cdot 9 - 3 \cdot 7) - 8(1 \cdot 6 - 3 \cdot 4) = 0.$$

Example. $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$.

Subtract the 1st row from the 2nd row and from the 3rd row:

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 6 & 6 & 6 \end{vmatrix} = 0$$

since the last matrix has two proportional rows.

Another example. $B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 13 \end{pmatrix}$.

Let's do some row reduction.

Add -4 times the 1st row to the 2nd row:

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 13 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 13 \end{vmatrix}$$

Add -7 times the 1st row to the 3rd row:

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 13 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -8 \end{vmatrix}$$

Expand the determinant by the 1st column:

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -8 \end{vmatrix} = 1 \begin{vmatrix} -3 & -6 \\ -6 & -8 \end{vmatrix}$$

Thus

$$\det B = \begin{vmatrix} -3 & -6 \\ -6 & -8 \end{vmatrix} = (-3) \begin{vmatrix} 1 & 2 \\ -6 & -8 \end{vmatrix}$$

$$= (-3)(-2) \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = (-3)(-2)(-2) = -12.$$

Example. $C = \begin{pmatrix} 2 & -2 & 0 & 3 \\ -5 & 3 & 2 & 1 \\ 1 & -1 & 0 & -3 \\ 2 & 0 & 0 & -1 \end{pmatrix}$, $\det C = ?$

Expand the determinant by the 3rd column:

$$\begin{vmatrix} 2 & -2 & 0 & 3 \\ -5 & 3 & 2 & 1 \\ 1 & -1 & 0 & -3 \\ 2 & 0 & 0 & -1 \end{vmatrix} = -2 \begin{vmatrix} 2 & -2 & 3 \\ 1 & -1 & -3 \\ 2 & 0 & -1 \end{vmatrix}$$

Add -2 times the 2nd row to the 1st row:

$$\det C = -2 \begin{vmatrix} 2 & -2 & 3 \\ 1 & -1 & -3 \\ 2 & 0 & -1 \end{vmatrix} = -2 \begin{vmatrix} 0 & 0 & 9 \\ 1 & -1 & -3 \\ 2 & 0 & -1 \end{vmatrix}$$

Expand the determinant by the 1st row:

$$\det C = -2 \begin{vmatrix} 0 & 0 & 9 \\ 1 & -1 & -3 \\ 2 & 0 & -1 \end{vmatrix} = -2 \cdot 9 \begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix}$$

Thus

$$\det C = -18 \begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix} = -18 \cdot 2 = -36.$$

Problem. For what values of a will the following system have a unique solution?

$$\begin{cases} x + 2y + z = 1 \\ -x + 4y + 2z = 2 \\ 2x - 2y + az = 3 \end{cases}$$

The system has a unique solution if and only if the coefficient matrix is invertible.

$$A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 4 & 2 \\ 2 & -2 & a \end{pmatrix}, \quad \det A = ?$$

Add -2 times the 3rd column to the 2nd column:

$$\begin{vmatrix} 1 & 2 & 1 \\ -1 & 4 & 2 \\ 2 & -2 & a \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ -1 & 0 & 2 \\ 2 & -2-2a & a \end{vmatrix}$$

Expand the determinant by the 2nd column:

$$\det A = \begin{vmatrix} 1 & 0 & 1 \\ -1 & 0 & 2 \\ 2 & -2-2a & a \end{vmatrix} = -(-2-2a) \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix}$$

Hence $\det A = -(-2-2a) \cdot 3 = 6(1+a)$.
Thus A is invertible if and only if $a \neq -1$.

More properties of determinants

Determinants and matrix multiplication:

- if A and B are $n \times n$ matrices then
$$\det(AB) = \det A \cdot \det B;$$
- if A and B are $n \times n$ matrices then
$$\det(AB) = \det(BA);$$
- if A is an invertible matrix then
$$\det(A^{-1}) = (\det A)^{-1}.$$

Determinants and scalar multiplication:

- if A is an $n \times n$ matrix and $r \in \mathbb{R}$ then
$$\det(rA) = r^n \det A.$$

Examples

$$X = \begin{pmatrix} -1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & -3 \end{pmatrix}, \quad Y = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 3 & 0 \\ 2 & -2 & 1 \end{pmatrix}.$$

$$\det X = (-1)\cdot 2\cdot(-3) = 6, \quad \det Y = \det Y^T = 3,$$

$$\det(XY) = 6 \cdot 3 = 18, \quad \det(YX) = 3 \cdot 6 = 18,$$

$$\det(Y^{-1}) = 1/3, \quad \det(XY^{-1}) = 6/3 = 2,$$

$$\det(XYX^{-1}) = \det Y = 3, \quad \det(X^{-1}Y^{-1}XY) = 1,$$

$$\det(2X) = 2^3 \det X = 2^3 \cdot 6 = 48,$$

$$\det(-3X^T XY^{-4}) = (-3)^3 \cdot 6 \cdot 6 \cdot 3^{-4} = -12.$$

The Vandermonde determinant

Definition. The **Vandermonde determinant** is the determinant of the following matrix

$$V = \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ 1 & x_3 & x_3^2 & \cdots & x_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{pmatrix},$$

where $x_1, x_2, \dots, x_n \in \mathbb{R}$. Equivalently, $V = (a_{ij})_{1 \le i,j \le n}$, where $a_{ij} = x_i^{j-1}$.

Examples.

- $\begin{vmatrix} 1 & x_1 \\ 1 & x_2 \end{vmatrix} = x_2 - x_1.$

- $\begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix} = \begin{vmatrix} 1 & x_1 & 0 \\ 1 & x_2 & x_2^2 - x_1 x_2 \\ 1 & x_3 & x_3^2 - x_1 x_3 \end{vmatrix}$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 1 & x_2 - x_1 & x_2^2 - x_1 x_2 \\ 1 & x_3 - x_1 & x_3^2 - x_1 x_3 \end{vmatrix} = \begin{vmatrix} x_2 - x_1 & x_2^2 - x_1 x_2 \\ x_3 - x_1 & x_3^2 - x_1 x_3 \end{vmatrix}$$

$$= (x_2 - x_1) \begin{vmatrix} 1 & x_2 \\ x_3 - x_1 & x_3^2 - x_1 x_3 \end{vmatrix} = (x_2 - x_1)(x_3 - x_1) \begin{vmatrix} 1 & x_2 \\ 1 & x_3 \end{vmatrix}$$

$$= (x_2 - x_1)(x_3 - x_1)(x_3 - x_2).$$

Theorem

$$\begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ 1 & x_3 & x_3^2 & \cdots & x_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{vmatrix} = \prod_{1 \le i < j \le n} (x_j - x_i).$$

Corollary The Vandermonde determinant is not equal to 0 if and only if the numbers $x_1, x_2, \ldots, x_n$ are distinct.

Let $x_1, x_2, \ldots, x_n$ be distinct real numbers.

Theorem For any $b_1, b_2, \ldots, b_n \in \mathbb{R}$ there exists a unique polynomial $p(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}$ of degree less than n such that $p(x_i) = b_i$, $1 \le i \le n$.

$$\begin{cases} a_0 + a_1 x_1 + a_2 x_1^2 + \cdots + a_{n-1} x_1^{n-1} = b_1 \\ a_0 + a_1 x_2 + a_2 x_2^2 + \cdots + a_{n-1} x_2^{n-1} = b_2 \\ \cdots\cdots\cdots\cdots \\ a_0 + a_1 x_n + a_2 x_n^2 + \cdots + a_{n-1} x_n^{n-1} = b_n \end{cases}$$

$a_0, a_1, \ldots, a_{n-1}$ are unknowns. The coefficient matrix is the Vandermonde matrix.

A general system of n linear equations in n variables:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases} \iff A\mathbf{x} = \mathbf{b},$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

Cramer's rule

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases} \iff A\mathbf{x} = \mathbf{b}$$

Theorem Assume that the matrix A is invertible. Then the only solution of the system is given by

$$x_i = \frac{\det A_i}{\det A}, \quad i = 1, 2, \ldots, n,$$

where the matrix A_i is obtained by substituting the vector $\mathbf{b}$ for the ith column of A.

Example.

$$\begin{cases} x + 2y + 3z = 0 \\ 4x + 5y + 6z = 0 \\ 7x + 8y + 13z = 1 \end{cases}$$

Augmented matrix of the system:

$$(A \,|\, \mathbf{b}) = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 13 & 1 \end{array}\right)$$

As obtained earlier in this lecture, $\det A = -12$.
Since $\det A \neq 0$, there exists a unique solution of the system.

Example.

$$\begin{cases} x + 2y + 3z = 0 \\ 4x + 5y + 6z = 0 \\ 7x + 8y + 13z = 1 \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 13 & 1 \end{array}\right)$$

By Cramer's rule,

$$x = \frac{\begin{vmatrix} 0 & 2 & 3 \\ 0 & 5 & 6 \\ 1 & 8 & 13 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 13 \end{vmatrix}} = \frac{\begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix}}{-12} = \frac{-3}{-12} = \frac{1}{4}$$

Example.

$$\begin{cases} x + 2y + 3z = 0 \\ 4x + 5y + 6z = 0 \\ 7x + 8y + 13z = 1 \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 13 & 1 \end{array}\right)$$

By Cramer's rule,

$$y = \frac{\begin{vmatrix} 1 & 0 & 3 \\ 4 & 0 & 6 \\ 7 & 1 & 13 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 13 \end{vmatrix}} = \frac{-\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix}}{-12} = \frac{6}{-12} = -\frac{1}{2}$$

Example.

$$\begin{cases} x + 2y + 3z = 0 \\ 4x + 5y + 6z = 0 \\ 7x + 8y + 13z = 1 \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 13 & 1 \end{array}\right)$$

By Cramer's rule,

$$z = \frac{\begin{vmatrix} 1 & 2 & 0 \\ 4 & 5 & 0 \\ 7 & 8 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 13 \end{vmatrix}} = \frac{\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}}{-12} = \frac{-3}{-12} = \frac{1}{4}$$

System of linear equations:

$$\begin{cases} x + 2y + 3z = 0 \\ 4x + 5y + 6z = 0 \\ 7x + 8y + 13z = 1 \end{cases}$$

Solution: $(x, y, z) = \left(\frac{1}{4}, -\frac{1}{2}, \frac{1}{4}\right).$

Determinants and the inverse matrix

Given an $n \times n$ matrix $A = (a_{ij})$, let M_{ij} denote the $(n-1)\times(n-1)$ submatrix obtained by deleting the ith row and the jth column of A.

The **cofactor matrix** of A is an $n \times n$ matrix $\widetilde{A} = (\alpha_{ij})$ defined by $\alpha_{ij} = (-1)^{i+j} \det M_{ij}$.

Theorem $\widetilde{A}^T A = A\widetilde{A}^T = (\det A)I$.

Corollary If $\det A \neq 0$ then the matrix A is invertible and $A^{-1} = (\det A)^{-1}\widetilde{A}^T$.

$A\widetilde{A}^T = (\det A)I$ means that

$$\sum_{j=1}^{n}(-1)^{k+j} a_{kj} \det M_{kj} = \det A \quad \text{for all } k,$$

$$\sum_{j=1}^{n}(-1)^{k+j} a_{mj} \det M_{kj} = 0 \quad \text{for } m \neq k.$$