MATH 304

Linear Algebra

Lecture 25:
Complex eigenvalues and eigenvectors.
Orthogonal matrices.
Rotations in space.

Complex numbers

$\mathbb{C}$: complex numbers.

Complex number: $\boxed{z = x + iy,}$

where $x, y \in \mathbb{R}$ and $i^2 = -1$.

$i = \sqrt{-1}$: imaginary unit

Alternative notation: $z = x + yi$.

$x =$ real part of z,
$iy =$ imaginary part of z

$y = 0 \implies z = x$ (real number)
$x = 0 \implies z = iy$ (purely imaginary number)

We add, subtract, and multiply complex numbers as polynomials in i (but keep in mind that $i^2 = -1$).

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2),$$
$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2),$$
$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1).$$

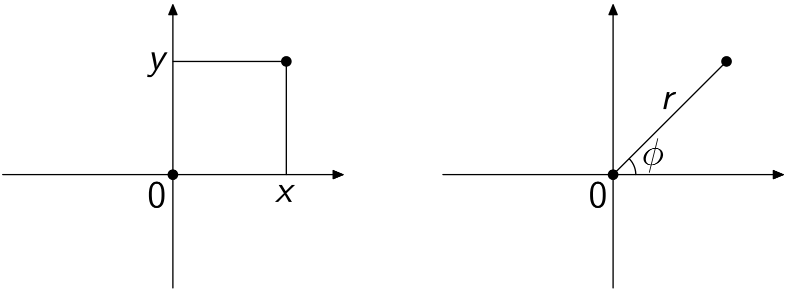
Given $z = x + iy$, the **complex conjugate** of z is $\bar{z} = x - iy$. The **modulus** of z is $|z| = \sqrt{x^2 + y^2}$.

$$z\bar{z} = (x + iy)(x - iy) = x^2 - (iy)^2 = x^2 + y^2 = |z|^2.$$

$$z^{-1} = \frac{\bar{z}}{|z|^2}, \qquad (x + iy)^{-1} = \frac{x - iy}{x^2 + y^2}.$$

Geometric representation

Any complex number $z = x + iy$ is represented by the vector/point $(x, y) \in \mathbb{R}^2$.



$$x = r\cos\phi,\ \ y = r\sin\phi \implies z = r(\cos\phi + i\sin\phi) = re^{i\phi}$$

If $z_1 = r_1 e^{i\phi_1}$ and $z_2 = r_2 e^{i\phi_2}$, then $z_1 z_2 = r_1 r_2 e^{i(\phi_1+\phi_2)}$, $z_1/z_2 = (r_1/r_2)e^{i(\phi_1-\phi_2)}$.

Fundamental Theorem of Algebra

Any polynomial of degree $n \geq 1$, with complex coefficients, has exactly n roots (counting with multiplicities).

Equivalently, if

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

where $a_i \in \mathbb{C}$ and $a_n \neq 0$, then there exist complex numbers $z_1, z_2, \ldots, z_n$ such that

$$p(z) = a_n (z - z_1)(z - z_2) \ldots (z - z_n).$$

Complex eigenvalues/eigenvectors

Example. $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. $\det(A - \lambda I) = \lambda^2 + 1$.

Characteristic roots: $\lambda_1 = i$ and $\lambda_2 = -i$.

Associated eigenvectors: $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}$.

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} i \\ 1 \end{pmatrix} = i \begin{pmatrix} 1 \\ -i \end{pmatrix},$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} -i \\ 1 \end{pmatrix} = -i \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

$\mathbf{v}_1, \mathbf{v}_2$ is a basis of eigenvectors. *In which space?*

Complexification

Instead of the real vector space $\mathbb{R}^2$, we consider a *complex vector space* $\mathbb{C}^2$ (all complex numbers are admissible as scalars).

The linear operator $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(\mathbf{x}) = A\mathbf{x}$ is extended to a *complex linear operator* $F : \mathbb{C}^2 \to \mathbb{C}^2$, $F(\mathbf{x}) = A\mathbf{x}$.

The vectors $\mathbf{v}_1 = (1, -i)$ and $\mathbf{v}_2 = (1, i)$ form a basis for $\mathbb{C}^2$.

$\mathbb{C}^2$ is also a real vector space (of real dimension 4). The standard real basis for $\mathbb{C}^2$ is $\mathbf{e}_1 = (1, 0)$, $\mathbf{e}_2 = (0, 1)$, $i\mathbf{e}_1 = (i, 0)$, $i\mathbf{e}_2 = (0, i)$. The matrix of the operator F with respect to this basis has the block structure $\begin{pmatrix} A & O \\ O & A \end{pmatrix}$.

Dot product of complex vectors

Dot product of real vectors
$\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$:

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

Dot product of complex vectors
$\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{C}^n$:

$$\mathbf{x} \cdot \mathbf{y} = x_1 \overline{y_1} + x_2 \overline{y_2} + \cdots + x_n \overline{y_n}.$$

If $z = r + it$ $(t, s \in \mathbb{R})$ then $\overline{z} = r - it$,
$z\overline{z} = r^2 + t^2 = |z|^2$.

Hence $\mathbf{x} \cdot \mathbf{x} = |x_1|^2 + |x_2|^2 + \cdots + |x_n|^2 \geq 0$.

Also, $\mathbf{x} \cdot \mathbf{x} = 0$ if and only if $\mathbf{x} = \mathbf{0}$.

The norm is defined by $\|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$.

Normal matrices

Definition. An $n \times n$ matrix A is called

- **symmetric** if $A^T = A$;
- **orthogonal** if $AA^T = A^T A = I$, i.e., $A^T = A^{-1}$;
- **normal** if $AA^T = A^T A$.

Theorem Let A be an $n \times n$ matrix with real entries. Then
(a) A is normal $\iff$ there exists an orthonormal basis for $\mathbb{C}^n$ consisting of eigenvectors of A;
(b) A is symmetric $\iff$ there exists an orthonormal basis for $\mathbb{R}^n$ consisting of eigenvectors of A.

Example. $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

- A is symmetric.
- A has three eigenvalues: 0, 2, and 3.
- Associated eigenvectors are $\mathbf{v}_1 = (-1, 0, 1)$, $\mathbf{v}_2 = (1, 0, 1)$, and $\mathbf{v}_3 = (0, 1, 0)$, respectively.
- Vectors $\frac{1}{\sqrt{2}}\mathbf{v}_1, \frac{1}{\sqrt{2}}\mathbf{v}_2, \mathbf{v}_3$ form an orthonormal basis for $\mathbb{R}^3$.

Theorem Suppose A is a normal matrix. Then for any $\mathbf{x} \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$ one has

$$A\mathbf{x} = \lambda\mathbf{x} \iff A^T\mathbf{x} = \overline{\lambda}\mathbf{x}.$$

Thus any normal matrix A shares with A^T all real eigenvalues and the corresponding eigenvectors. Also, $A\mathbf{x} = \lambda\mathbf{x} \iff A\overline{\mathbf{x}} = \overline{\lambda}\,\overline{\mathbf{x}}$ for any matrix A with real entries.

Corollary All eigenvalues λ of a symmetric matrix are real ($\overline{\lambda} = \lambda$). All eigenvalues λ of an orthogonal matrix satisfy $\overline{\lambda} = \lambda^{-1} \iff |\lambda| = 1$.

Why are orthogonal matrices called so?

Theorem Given an $n \times n$ matrix A, the following conditions are equivalent:
(i) A is orthogonal: $A^T = A^{-1}$;
(ii) columns of A form an orthonormal basis for $\mathbb{R}^n$;
(iii) rows of A form an orthonormal basis for $\mathbb{R}^n$.

Proof: Entries of the matrix $A^T A$ are dot products of columns of A. Entries of AA^T are dot products of rows of A.

In particular, an orthogonal matrix is the transition matrix from one orthonormal basis to another.

Example. $A_\phi = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$.

- $A_\phi A_\psi = A_{\phi+\psi}$
- $A_\phi^{-1} = A_{-\phi} = A_\phi^T$
- A_ϕ is orthogonal
- $\det(A_\phi - \lambda I) = (\cos\phi - \lambda)^2 + \sin^2\phi$.
- Eigenvalues: $\lambda_1 = \cos\phi + i\sin\phi = e^{i\phi}$, $\lambda_2 = \cos\phi - i\sin\phi = e^{-i\phi}$.
- Associated eigenvectors: $\mathbf{v}_1 = (1, -i)$, $\mathbf{v}_2 = (1, i)$.
- Vectors $\frac{1}{\sqrt{2}}\mathbf{v}_1$ and $\frac{1}{\sqrt{2}}\mathbf{v}_2$ form an orthonormal basis for $\mathbb{C}^2$.

Consider a linear operator $L : \mathbb{R}^n \to \mathbb{R}^n$, $L(\mathbf{x}) = A\mathbf{x}$, where A is an $n \times n$ matrix.

Theorem The following conditions are equivalent:
(i) $|L(\mathbf{x})| = |\mathbf{x}|$ for all $\mathbf{x} \in \mathbb{R}^n$;
(ii) $L(\mathbf{x}) \cdot L(\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$;
(iii) the matrix A is orthogonal.

Definition. A transformation $f : \mathbb{R}^n \to \mathbb{R}^n$ is called an **isometry** if it preserves distances between points: $|f(\mathbf{x}) - f(\mathbf{y})| = |\mathbf{x} - \mathbf{y}|$.

Theorem Any isometry $f : \mathbb{R}^n \to \mathbb{R}^n$ can be represented as $f(\mathbf{x}) = A\mathbf{x} + \mathbf{x}_0$, where $\mathbf{x}_0 \in \mathbb{R}^n$ and A is an orthogonal matrix.

Consider a linear operator $L : \mathbb{R}^n \to \mathbb{R}^n$, $L(\mathbf{x}) = A\mathbf{x}$, where A is an $n \times n$ orthogonal matrix.

Theorem There exists an orthonormal basis for $\mathbb{R}^n$ such that the matrix of L relative to this basis has the diagonal block structure

$$\begin{pmatrix} D_{\pm 1} & O & \dots & O \\ O & R_1 & \dots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \dots & R_k \end{pmatrix},$$

where $D_{\pm 1}$ is a diagonal matrix whose diagonal entries are equal to 1 or -1, and

$$R_j = \begin{pmatrix} \cos\phi_j & -\sin\phi_j \\ \sin\phi_j & \cos\phi_j \end{pmatrix}, \quad \phi_j \in \mathbb{R}.$$

Classification of 2×2 orthogonal matrices:

	$\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
	rotation about the origin	reflection in a line
Determinant:	1	-1
Eigenvalues:	$e^{i\phi}$ and $e^{-i\phi}$	-1 and 1

Classification of 3×3 orthogonal matrices:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

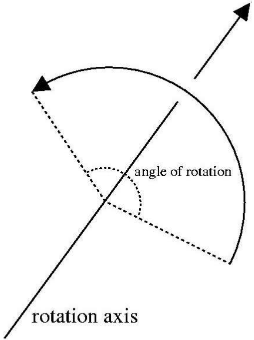
$$C = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}.$$

$A =$ rotation about a line; $B =$ reflection in a plane; $C =$ rotation about a line combined with reflection in the orthogonal plane.

$\det A = 1$, $\det B = \det C = -1$.

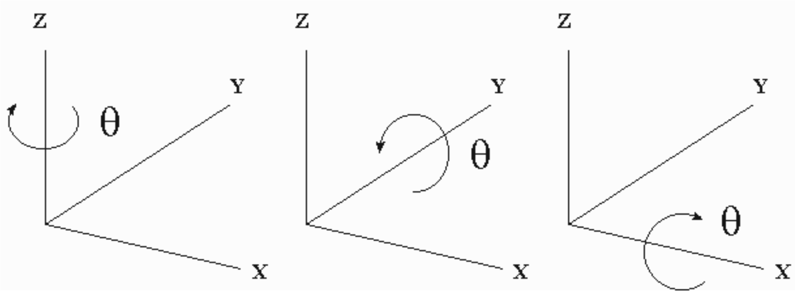
A has eigenvalues $1, e^{i\phi}, e^{-i\phi}$. B has eigenvalues $-1, 1, 1$. C has eigenvalues $-1, e^{i\phi}, e^{-i\phi}$.

Rotations in space



If the axis of rotation is oriented, we can say about *clockwise* or *counterclockwise* rotations (with respect to the view from the positive semi-axis).

Clockwise rotations about coordinate axes



$$
\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}
\begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}
\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}
$$

Problem. Find the matrix of the rotation by 90° about the line spanned by the vector $\mathbf{a} = (1, 2, 2)$. The rotation is assumed to be counterclockwise when looking from the tip of **a**.

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$ is the matrix of (counterclockwise) rotation by 90° about the x-axis.

We need to find an orthonormal basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ such that $\mathbf{v}_1$ points in the same direction as **a**. Also, the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ should obey the same hand rule as the standard basis. Then B will be the matrix of the given rotation relative to the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

Let U denote the transition matrix from the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to the standard basis (columns of U are vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$). Then the desired matrix is $A = UBU^{-1}$.

Since $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is going to be an orthonormal basis, the matrix U will be orthogonal. Then $U^{-1} = U^T$ and $A = UBU^T$.

Remark. The basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ obeys the same hand rule as the standard basis if and only if $\det U > 0$.

Hint. Vectors $\mathbf{a} = (1, 2, 2)$, $\mathbf{b} = (-2, -1, 2)$, and $\mathbf{c} = (2, -2, 1)$ are orthogonal.

We have $|\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}| = 3$, hence $\mathbf{v}_1 = \frac{1}{3}\mathbf{a}$, $\mathbf{v}_2 = \frac{1}{3}\mathbf{b}$, $\mathbf{v}_3 = \frac{1}{3}\mathbf{c}$ is an orthonormal basis.

Transition matrix: $U = \frac{1}{3}\begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{pmatrix}$.

$$\det U = \frac{1}{27}\begin{vmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{vmatrix} = \frac{1}{27} \cdot 27 = 1.$$

In the case $\det U = -1$, we would change $\mathbf{v}_3$ to $-\mathbf{v}_3$, or change $\mathbf{v}_2$ to $-\mathbf{v}_2$, or interchange $\mathbf{v}_2$ and $\mathbf{v}_3$.

$$A = UBU^T$$

$$= \tfrac{1}{3}\begin{pmatrix}1 & -2 & 2\\ 2 & -1 & -2\\ 2 & 2 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{pmatrix}\cdot\tfrac{1}{3}\begin{pmatrix}1 & 2 & 2\\ -2 & -1 & 2\\ 2 & -2 & 1\end{pmatrix}$$

$$= \tfrac{1}{9}\begin{pmatrix}1 & 2 & 2\\ 2 & -2 & 1\\ 2 & 1 & -2\end{pmatrix}\begin{pmatrix}1 & 2 & 2\\ -2 & -1 & 2\\ 2 & -2 & 1\end{pmatrix}$$

$$= \tfrac{1}{9}\begin{pmatrix}1 & -4 & 8\\ 8 & 4 & 1\\ -4 & 7 & 4\end{pmatrix}.$$

$U = \frac{1}{3}\begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{pmatrix}$ is an orthogonal matrix.

$\det U = 1 \implies U$ is a rotation matrix.

Problem. (a) Find the axis of the rotation.
(b) Find the angle of the rotation.

The axis is the set of points $\mathbf{x} \in \mathbb{R}^n$ such that $U\mathbf{x} = \mathbf{x} \iff (U - I)\mathbf{x} = \mathbf{0}$. To find the axis, we apply row reduction to the matrix

$$3(U - I) = 3U - 3I = \begin{pmatrix} -2 & -2 & 2 \\ 2 & -4 & -2 \\ 2 & 2 & -2 \end{pmatrix}.$$

$$\begin{pmatrix} -2 & -2 & 2 \\ 2 & -4 & -2 \\ 2 & 2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 \\ 2 & -4 & -2 \\ 2 & 2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 \\ 0 & -6 & 0 \\ 2 & 2 & -2 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & -1 \\ 0 & -6 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus $U\mathbf{x} = \mathbf{x} \iff \begin{cases} x - z = 0, \\ y = 0. \end{cases}$

The general solution is $x = t,\ y = 0,\ z = t,$ where $t \in \mathbb{R}$.

$\Longrightarrow$ $\mathbf{d} = (1, 0, 1)$ is the direction of the axis.

$$U = \frac{1}{3}\begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{pmatrix}$$

Let ϕ be the angle of rotation. Then the eigenvalues of U are 1, $e^{i\phi}$, and $e^{-i\phi}$. Therefore

$$\det(U - \lambda I) = (1 - \lambda)(e^{i\phi} - \lambda)(e^{-i\phi} - \lambda).$$

Besides, $\det(U - \lambda I) = -\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3$, where $c_1 = \mathrm{tr}\, U$ (the sum of diagonal entries). It follows that

$$\mathrm{tr}\, U = 1 + e^{i\phi} + e^{-i\phi} = 1 + 2\cos\phi.$$

$$\mathrm{tr}\, U = 1/3 \implies \cos\phi = -1/3 \implies \phi \approx 109.47^{\mathrm{o}}$$