

Sample problems for Test 2: Solutions

Any problem may be altered or replaced by a different one!

Problem 1 (15 pts.) Let $\mathcal{M}_{2,2}(\mathbb{R})$ denote the vector space of 2×2 matrices with real entries. Consider a linear operator $L : \mathcal{M}_{2,2}(\mathbb{R}) \rightarrow \mathcal{M}_{2,2}(\mathbb{R})$ given by

$$L \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

Find the matrix of the operator L with respect to the basis

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let M_L denote the desired matrix. By definition, M_L is a 4×4 matrix whose columns are coordinates of the matrices $L(E_1), L(E_2), L(E_3), L(E_4)$ with respect to the basis E_1, E_2, E_3, E_4 . We have that

$$L(E_1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} = 1E_1 + 2E_2 + 0E_3 + 0E_4,$$

$$L(E_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 0 & 0 \end{pmatrix} = 3E_1 + 4E_2 + 0E_3 + 0E_4,$$

$$L(E_3) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} = 0E_1 + 0E_2 + 1E_3 + 2E_4,$$

$$L(E_4) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 3 & 4 \end{pmatrix} = 0E_1 + 0E_2 + 3E_3 + 4E_4.$$

It follows that

$$M_L = \begin{pmatrix} 1 & 3 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 4 \end{pmatrix}.$$

Problem 2 (20 pts.) Find a linear polynomial which is the best least squares fit to the following data:

$$\frac{x}{f(x)} \begin{array}{c|c|c|c|c|c} -2 & -1 & 0 & 1 & 2 \\ \hline -3 & -2 & 1 & 2 & 5 \end{array}$$

We are looking for a function $f(x) = c_1 + c_2x$, where c_1, c_2 are unknown coefficients. The data of the problem give rise to an overdetermined system of linear equations in variables c_1 and c_2 :

$$\begin{cases} c_1 - 2c_2 = -3, \\ c_1 - c_2 = -2, \\ c_1 = 1, \\ c_1 + c_2 = 2, \\ c_1 + 2c_2 = 5. \end{cases}$$

This system is inconsistent. We can represent it as a matrix equation $\mathbf{A}\mathbf{c} = \mathbf{y}$, where

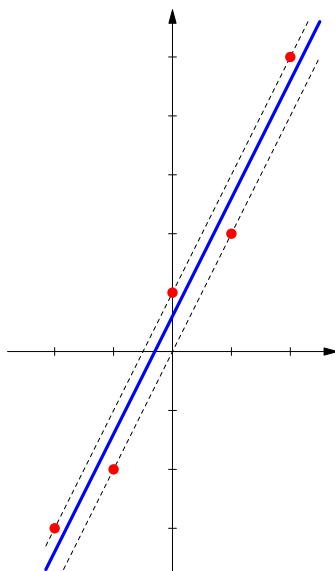
$$\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} -3 \\ -2 \\ 1 \\ 2 \\ 5 \end{pmatrix}.$$

The least squares solution \mathbf{c} of the above system is a solution of the system $\mathbf{A}^T\mathbf{A}\mathbf{c} = \mathbf{A}^T\mathbf{y}$:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} -3 \\ -2 \\ 1 \\ 2 \\ 5 \end{pmatrix}$$

$$\iff \begin{pmatrix} 5 & 0 \\ 0 & 10 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 20 \end{pmatrix} \iff \begin{cases} c_1 = 3/5 \\ c_2 = 2 \end{cases}$$

Thus the function $f(x) = \frac{3}{5} + 2x$ is the best least squares fit to the above data among linear polynomials.



Problem 3 (25 pts.) Let V be a subspace of \mathbb{R}^4 spanned by the vectors $\mathbf{x}_1 = (1, 1, 1, 1)$ and $\mathbf{x}_2 = (1, 0, 3, 0)$.

(i) Find an orthonormal basis for V .

First we apply the Gram-Schmidt orthogonalization process to vectors $\mathbf{x}_1, \mathbf{x}_2$ and obtain an orthogonal basis $\mathbf{v}_1, \mathbf{v}_2$ for the subspace V :

$$\mathbf{v}_1 = \mathbf{x}_1 = (1, 1, 1, 1), \quad \mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 = (1, 0, 3, 0) - \frac{4}{4}(1, 1, 1, 1) = (0, -1, 2, -1).$$

Then we normalize vectors $\mathbf{v}_1, \mathbf{v}_2$ to obtain an orthonormal basis $\mathbf{w}_1, \mathbf{w}_2$ for V :

$$\mathbf{w}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{2}\mathbf{v}_1 = \frac{1}{2}(1, 1, 1, 1), \quad \mathbf{w}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{1}{\sqrt{6}}\mathbf{v}_2 = \frac{1}{\sqrt{6}}(0, -1, 2, -1).$$

(ii) Find an orthonormal basis for the orthogonal complement V^\perp .

Since the subspace V is spanned by vectors $(1, 1, 1, 1)$ and $(1, 0, 3, 0)$, it is the row space of the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 3 & 0 \end{pmatrix}.$$

Then the orthogonal complement V^\perp is the nullspace of A . To find the nullspace, we convert the matrix A to reduced row echelon form:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 3 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 3 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 1 \end{pmatrix}.$$

Hence a vector $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ belongs to V^\perp if and only if

$$\begin{cases} x_1 + 3x_3 = 0 \\ x_2 - 2x_3 + x_4 = 0 \end{cases} \iff \begin{cases} x_1 = -3x_3 \\ x_2 = 2x_3 - x_4 \end{cases}$$

The general solution of the system is $(x_1, x_2, x_3, x_4) = (-3t, 2t - s, t, s) = t(-3, 2, 1, 0) + s(0, -1, 0, 1)$, where $t, s \in \mathbb{R}$. It follows that V^\perp is spanned by vectors $\mathbf{x}_3 = (0, -1, 0, 1)$ and $\mathbf{x}_4 = (-3, 2, 1, 0)$. It remains to orthogonalize and normalize this basis for V^\perp :

$$\mathbf{v}_3 = \mathbf{x}_3 = (0, -1, 0, 1), \quad \mathbf{v}_4 = \mathbf{x}_4 - \frac{\mathbf{x}_4 \cdot \mathbf{v}_3}{\mathbf{v}_3 \cdot \mathbf{v}_3} \mathbf{v}_3 = (-3, 2, 1, 0) - \frac{-2}{2}(0, -1, 0, 1) = (-3, 1, 1, 1),$$

$$\mathbf{w}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \frac{1}{\sqrt{2}}(0, -1, 0, 1), \quad \mathbf{w}_4 = \frac{\mathbf{v}_4}{\|\mathbf{v}_4\|} = \frac{1}{2\sqrt{3}}\mathbf{v}_4 = \frac{1}{2\sqrt{3}}(-3, 1, 1, 1).$$

Thus the vectors $\mathbf{w}_3 = \frac{1}{\sqrt{2}}(0, -1, 0, 1)$ and $\mathbf{w}_4 = \frac{1}{2\sqrt{3}}(-3, 1, 1, 1)$ form an orthonormal basis for V^\perp .

Alternative solution: Suppose that an orthonormal basis $\mathbf{w}_1, \mathbf{w}_2$ for the subspace V has been extended to an orthonormal basis $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4$ for \mathbb{R}^4 . Then the vectors $\mathbf{w}_3, \mathbf{w}_4$ form an orthonormal basis for the orthogonal complement V^\perp .

We know that vectors $\mathbf{v}_1 = (1, 1, 1, 1)$ and $\mathbf{v}_2 = (0, -1, 2, -1)$ form an orthogonal basis for V . This basis can be extended to a basis for \mathbb{R}^4 by adding two vectors from the standard basis. For example, we can add vectors $\mathbf{e}_3 = (0, 0, 1, 0)$ and $\mathbf{e}_4 = (0, 0, 0, 1)$. The vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_3, \mathbf{e}_4$ do form a basis for \mathbb{R}^4 since the matrix whose rows are these vectors is nonsingular:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = -1 \neq 0.$$

To orthogonalize the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_3, \mathbf{e}_4$, we apply the Gram-Schmidt process (note that the vectors \mathbf{v}_1 and \mathbf{v}_2 are already orthogonal):

$$\mathbf{v}_3 = \mathbf{e}_3 - \frac{\mathbf{e}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 - \frac{\mathbf{e}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2} \mathbf{v}_2 = (0, 0, 1, 0) - \frac{1}{4}(1, 1, 1, 1) - \frac{2}{6}(0, -1, 2, -1) = \frac{1}{12}(-3, 1, 1, 1),$$

$$\begin{aligned} \mathbf{v}_4 &= \mathbf{e}_4 - \frac{\mathbf{e}_4 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 - \frac{\mathbf{e}_4 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2} \mathbf{v}_2 - \frac{\mathbf{e}_4 \cdot \mathbf{v}_3}{\mathbf{v}_3 \cdot \mathbf{v}_3} \mathbf{v}_3 = \\ &= (0, 0, 0, 1) - \frac{1}{4}(1, 1, 1, 1) - \frac{-1}{6}(0, -1, 2, -1) - \frac{1/12}{1/12} \cdot \frac{1}{12}(-3, 1, 1, 1) = \frac{1}{2}(0, -1, 0, 1). \end{aligned}$$

It remains to normalize vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$:

$$\mathbf{w}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{2}(1, 1, 1, 1), \quad \mathbf{w}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{1}{\sqrt{6}}(0, -1, 2, -1),$$

$$\mathbf{w}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \sqrt{12} \mathbf{v}_3 = \frac{1}{2\sqrt{3}}(-3, 1, 1, 1), \quad \mathbf{w}_4 = \frac{\mathbf{v}_4}{\|\mathbf{v}_4\|} = \sqrt{2} \mathbf{v}_4 = \frac{1}{\sqrt{2}}(0, -1, 0, 1).$$

We have obtained an orthonormal basis $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4$ for \mathbb{R}^4 that extends an orthonormal basis $\mathbf{w}_1, \mathbf{w}_2$ for the subspace V . It follows that $\mathbf{w}_3 = \frac{1}{2\sqrt{3}}(-3, 1, 1, 1)$, $\mathbf{w}_4 = \frac{1}{\sqrt{2}}(0, -1, 0, 1)$ is an orthonormal basis for V^\perp .

Problem 4 (30 pts.) Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$.

(i) Find all eigenvalues of the matrix A .

The eigenvalues of A are roots of the characteristic equation $\det(A - \lambda I) = 0$. We obtain that

$$\det(A - \lambda I) = \begin{vmatrix} 1 - \lambda & 2 & 0 \\ 1 & 1 - \lambda & 1 \\ 0 & 2 & 1 - \lambda \end{vmatrix} = (1 - \lambda)^3 - 2(1 - \lambda) - 2(1 - \lambda)$$

$$= (1 - \lambda)((1 - \lambda)^2 - 4) = (1 - \lambda)((1 - \lambda) - 2)((1 - \lambda) + 2) = -(\lambda - 1)(\lambda + 1)(\lambda - 3).$$

Hence the matrix A has three eigenvalues: -1 , 1 , and 3 .

(ii) For each eigenvalue of A , find an associated eigenvector.

An eigenvector $\mathbf{v} = (x, y, z)$ of A associated with an eigenvalue λ is a nonzero solution of the vector equation $(A - \lambda I)\mathbf{v} = \mathbf{0}$. To solve the equation, we apply row reduction to the matrix $A - \lambda I$.

First consider the case $\lambda = -1$. The row reduction yields

$$A + I = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 2 & 1 \\ 0 & 2 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 2 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence

$$(A + I)\mathbf{v} = \mathbf{0} \iff \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{cases} x - z = 0, \\ y + z = 0. \end{cases}$$

The general solution is $x = t$, $y = -t$, $z = t$, where $t \in \mathbb{R}$. In particular, $\mathbf{v}_1 = (1, -1, 1)$ is an eigenvector of A associated with the eigenvalue -1 .

Secondly, consider the case $\lambda = 1$. The row reduction yields

$$A - I = \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence

$$(A - I)\mathbf{v} = \mathbf{0} \iff \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{cases} x + z = 0, \\ y = 0. \end{cases}$$

The general solution is $x = -t$, $y = 0$, $z = t$, where $t \in \mathbb{R}$. In particular, $\mathbf{v}_2 = (-1, 0, 1)$ is an eigenvector of A associated with the eigenvalue 1.

Finally, consider the case $\lambda = 3$. The row reduction yields

$$\begin{aligned} A - 3I &= \begin{pmatrix} -2 & 2 & 0 \\ 1 & -2 & 1 \\ 0 & 2 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 0 \\ 1 & -2 & 1 \\ 0 & 2 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 2 & -2 \end{pmatrix} \\ &\rightarrow \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Hence

$$(A - 3I)\mathbf{v} = \mathbf{0} \iff \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{cases} x - z = 0, \\ y - z = 0. \end{cases}$$

The general solution is $x = t$, $y = t$, $z = t$, where $t \in \mathbb{R}$. In particular, $\mathbf{v}_3 = (1, 1, 1)$ is an eigenvector of A associated with the eigenvalue 3.

(iii) Is the matrix A diagonalizable? Explain.

The matrix A is diagonalizable, i.e., there exists a basis for \mathbb{R}^3 formed by its eigenvectors. Namely, the vectors $\mathbf{v}_1 = (1, -1, 1)$, $\mathbf{v}_2 = (-1, 0, 1)$, and $\mathbf{v}_3 = (1, 1, 1)$ are eigenvectors of the matrix A belonging to distinct eigenvalues. Therefore these vectors are linearly independent. It follows that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is a basis for \mathbb{R}^3 .

Alternatively, the existence of a basis for \mathbb{R}^3 consisting of eigenvectors of A already follows from the fact that the matrix A has three distinct eigenvalues.

(iv) Find all eigenvalues of the matrix A^2 .

Suppose that \mathbf{v} is an eigenvector of the matrix A associated with an eigenvalue λ , that is, $\mathbf{v} \neq \mathbf{0}$ and $A\mathbf{v} = \lambda\mathbf{v}$. Then

$$A^2\mathbf{v} = A(A\mathbf{v}) = A(\lambda\mathbf{v}) = \lambda(A\mathbf{v}) = \lambda(\lambda\mathbf{v}) = \lambda^2\mathbf{v}.$$

Therefore \mathbf{v} is also an eigenvector of the matrix A^2 and the associated eigenvalue is λ^2 . We already know that the matrix A has eigenvalues -1 , 1 , and 3 . It follows that A^2 has eigenvalues 1 and 9 .

Since a 3×3 matrix can have up to 3 eigenvalues, we need an additional argument to show that 1 and 9 are the only eigenvalues of A^2 . The matrix A is diagonalizable. Namely, $A = UBU^{-1}$, where

$$B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

and U is the matrix whose columns are eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$:

$$U = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Then $A^2 = UBU^{-1}UBU^{-1} = UB^2U^{-1}$. It follows that

$$\det(A^2 - \lambda I) = \det(UB^2U^{-1} - \lambda I) = \det(UB^2U^{-1} - U(\lambda I)U^{-1})$$

$$= \det(U(B^2 - \lambda I)U^{-1}) = \det(U) \det(B^2 - \lambda I) \det(U^{-1}) = \det(B^2 - \lambda I).$$

Thus the matrix A^2 has the same characteristic polynomial as the diagonal matrix

$$B^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{pmatrix}.$$

Consequently, the matrices A^2 and B^2 have the same eigenvalues. The latter has eigenvalues 1 and 9.

Bonus Problem 5 (15 pts.) Let $L : V \rightarrow W$ be a linear mapping of a finite-dimensional vector space V to a vector space W . Show that

$$\dim \text{Range}(L) + \dim \ker(L) = \dim V.$$

The kernel $\ker(L)$ is a subspace of V . Since the vector space V is finite-dimensional, so is $\ker(L)$. Take a basis $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ for the subspace $\ker(L)$, then extend it to a basis $\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$ for the entire space V . We are going to prove that vectors $L(\mathbf{u}_1), L(\mathbf{u}_2), \dots, L(\mathbf{u}_m)$ form a basis for the range $L(V)$. Then $\dim \text{Range}(L) = m$, $\dim \ker(L) = k$, and $\dim V = k + m$.

Spanning: Any vector $\mathbf{w} \in \text{Range}(L)$ is represented as $\mathbf{w} = L(\mathbf{v})$, where $\mathbf{v} \in V$. We have

$$\mathbf{v} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \dots + \alpha_k \mathbf{v}_k + \beta_1 \mathbf{u}_1 + \beta_2 \mathbf{u}_2 + \dots + \beta_m \mathbf{u}_m$$

for some $\alpha_i, \beta_j \in \mathbb{R}$. It follows that

$$\mathbf{w} = L(\mathbf{v}) = \alpha_1 L(\mathbf{v}_1) + \dots + \alpha_k L(\mathbf{v}_k) + \beta_1 L(\mathbf{u}_1) + \dots + \beta_m L(\mathbf{u}_m) = \beta_1 L(\mathbf{u}_1) + \dots + \beta_m L(\mathbf{u}_m)$$

($L(\mathbf{v}_i) = \mathbf{0}$ since $\mathbf{v}_i \in \ker(L)$). Thus $\text{Range}(L)$ is spanned by the vectors $L(\mathbf{u}_1), L(\mathbf{u}_2), \dots, L(\mathbf{u}_m)$.

Linear independence: Suppose that $t_1 L(\mathbf{u}_1) + t_2 L(\mathbf{u}_2) + \dots + t_m L(\mathbf{u}_m) = \mathbf{0}$ for some $t_i \in \mathbb{R}$. Let $\mathbf{u} = t_1 \mathbf{u}_1 + t_2 \mathbf{u}_2 + \dots + t_m \mathbf{u}_m$. Since

$$L(\mathbf{u}) = t_1 L(\mathbf{u}_1) + t_2 L(\mathbf{u}_2) + \dots + t_m L(\mathbf{u}_m) = \mathbf{0},$$

the vector \mathbf{u} belongs to the kernel of L . Therefore $\mathbf{u} = s_1 \mathbf{v}_1 + s_2 \mathbf{v}_2 + \dots + s_k \mathbf{v}_k$ for some $s_j \in \mathbb{R}$. It follows that

$$t_1 \mathbf{u}_1 + t_2 \mathbf{u}_2 + \dots + t_m \mathbf{u}_m - s_1 \mathbf{v}_1 - s_2 \mathbf{v}_2 - \dots - s_k \mathbf{v}_k = \mathbf{u} - \mathbf{u} = \mathbf{0}.$$

Linear independence of vectors $\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{u}_1, \dots, \mathbf{u}_m$ implies that $t_1 = t_2 = \dots = t_m = 0$ (as well as $s_1 = s_2 = \dots = s_k = 0$). Thus the vectors $L(\mathbf{u}_1), L(\mathbf{u}_2), \dots, L(\mathbf{u}_m)$ are linearly independent.