MATH 304

Linear Algebra

Lecture 33: Diagonalization (continued).

Diagonalization

Let L be a linear operator on a finite-dimensional vector space V. Then the following conditions are equivalent:

- the matrix of L with respect to some basis is diagonal;
- there exists a basis for V formed by eigenvectors of L.

The operator L is **diagonalizable** if it satisfies these conditions.

Let A be an $n \times n$ matrix. Then the following conditions are equivalent:

- A is the matrix of a diagonalizable operator;
- A is similar to a diagonal matrix, i.e., it is represented as $A = UBU^{-1}$, where the matrix B is diagonal;
- there exists a basis for $\mathbb{R}^n$ formed by eigenvectors of A.

The matrix A is **diagonalizable** if it satisfies these conditions. Otherwise A is called **defective**.

To *diagonalize* an $n \times n$ matrix A is to find a diagonal matrix B and an invertible matrix U such that $A = UBU^{-1}$.

Suppose there exists a basis $\mathbf{v}_1, \dots, \mathbf{v}_n$ for $\mathbb{R}^n$ consisting of eigenvectors of A. That is, $A\mathbf{v}_k = \lambda_k \mathbf{v}_k$, where $\lambda_k \in \mathbb{R}$.
Then $A = UBU^{-1}$, where $B = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ and U is a transition matrix whose columns are vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$.

Example. $A = \begin{pmatrix} 4 & 3 \\ 0 & 1 \end{pmatrix}$.

Eigenvalues: $\lambda_1 = 4$, $\lambda_2 = 1$.
Associated eigenvectors: $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

Thus $A = UBU^{-1}$, where

$$B = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, \qquad U = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Matrix polynomials

Definition. Given an $n \times n$ matrix A, we let

$$A^2 = AA, \quad A^3 = AAA, \quad \dots, \quad A^k = \underbrace{AA \dots A}_{k \text{ times}}, \quad \dots$$

Also, let $A^1 = A$ and $A^0 = I_n$.

Associativity of matrix multiplication implies that all powers A^k are well defined and $A^j A^k = A^{j+k}$ for all $j, k \geq 0$. In particular, all powers of A commute.

Definition. For any polynomial

$$p(x) = c_0 x^m + c_1 x^{m-1} + \cdots + c_{m-1} x + c_m,$$

let $p(A) = c_0 A^m + c_1 A^{m-1} + \cdots + c_{m-1} A + c_m I_n$.

Theorem If $A = \mathrm{diag}(a_1, a_2, \dots, a_n)$, then $p(A) = \mathrm{diag}\big(p(a_1), p(a_2), \dots, p(a_n)\big)$.

Now suppose that the matrix A is diagonalizable. Then $A = UBU^{-1}$ for some diagonal matrix B and an invertible matrix U.

$A^2 = UBU^{-1}UBU^{-1} = UB^2U^{-1}$,
$A^3 = A^2A = UB^2U^{-1}UBU^{-1} = UB^3U^{-1}$.
Likewise, $A^n = UB^nU^{-1}$ for any $n \geq 1$.

$I + 2A - 3A^2 = UIU^{-1} + 2UBU^{-1} - 3UB^2U^{-1}$
$= U(I + 2B - 3B^2)U^{-1}$.
Likewise, $p(A) = Up(B)U^{-1}$ for any polynomial $p(x)$.

Problem. Let $A = \begin{pmatrix} 4 & 3 \\ 0 & 1 \end{pmatrix}$. Find a matrix C such that $C^2 = A$.

We know that $A = UBU^{-1}$, where

$$B = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, \qquad U = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Suppose that $D^2 = B$ for some matrix D. Let $C = UDU^{-1}$.
Then $C^2 = UDU^{-1}UDU^{-1} = UD^2U^{-1} = UBU^{-1} = A$.

We can take $D = \begin{pmatrix} \sqrt{4} & 0 \\ 0 & \sqrt{1} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$.

Then $C = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$.

Initial value problem for a system of linear ODEs:

$$\begin{cases} \frac{dx}{dt} = 4x + 3y, \\ \frac{dy}{dt} = y, \end{cases} \qquad x(0) = 1, \;\; y(0) = 1.$$

The system can be rewritten in vector form:

$$\frac{d\mathbf{v}}{dt} = A\mathbf{v}, \quad \text{where } \; A = \begin{pmatrix} 4 & 3 \\ 0 & 1 \end{pmatrix}, \;\; \mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Matrix A is diagonalizable: $A = UBU^{-1}$, where

$$B = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, \qquad U = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Let $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ be coordinates of the vector $\mathbf{v}$ relative to the basis $\mathbf{v}_1 = (1, 0)$, $\mathbf{v}_2 = (-1, 1)$ of eigenvectors of A. Then $\mathbf{v} = U\mathbf{w} \implies \mathbf{w} = U^{-1}\mathbf{v}$.

It follows that

$$\frac{d\mathbf{w}}{dt} = \frac{d}{dt}(U^{-1}\mathbf{v}) = U^{-1}\frac{d\mathbf{v}}{dt} = U^{-1}A\mathbf{v} = U^{-1}AU\mathbf{w}.$$

Hence $\quad \dfrac{d\mathbf{w}}{dt} = B\mathbf{w} \quad \Longleftrightarrow \quad \begin{cases} \frac{dw_1}{dt} = 4w_1, \\ \frac{dw_2}{dt} = w_2. \end{cases}$

General solution: $w_1(t) = c_1 e^{4t}$, $w_2(t) = c_2 e^t$, where $c_1, c_2 \in \mathbb{R}$.

Initial condition:

$$\mathbf{w}(0) = U^{-1}\mathbf{v}(0) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Thus $w_1(t) = 2e^{4t}$, $w_2(t) = e^t$. Then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = U\mathbf{w}(t) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2e^{4t} \\ e^t \end{pmatrix} = \begin{pmatrix} 2e^{4t} - e^t \\ e^t \end{pmatrix}.$$

- *Initial value problem for a linear ODE:*

$$\frac{dy}{dt} = 2y, \quad y(0) = 3.$$

Solution: $y(t) = 3e^{2t}$.

- *Initial value problem for a system of linear ODEs:*

$$\begin{cases} \frac{dx}{dt} = 2x + 3y, \\ \frac{dy}{dt} = x + 4y, \end{cases} \qquad x(0) = 2, \; y(0) = 1.$$

The system can be rewritten in vector form

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}, \text{ where } A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}.$$

Solution: $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{tA}\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$

What is e^{tA}*?*

Fibonacci numbers

The *Fibonacci numbers* are a sequence of integers $f_1, f_2, f_3, \ldots$ defined recursively by $f_1 = f_2 = 1$, $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$.

$$\boxed{1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ 233, \ldots}$$

Problem. Find $\displaystyle\lim_{n\to\infty} \frac{f_{n+1}}{f_n}$.

For any integer $n \geq 1$ let $\mathbf{v}_n = \begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix}$. Then

$$\begin{pmatrix} f_{n+2} \\ f_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix}.$$

That is, $\mathbf{v}_{n+1} = A\mathbf{v}_n$, where $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

In particular, $\mathbf{v}_2 = A\mathbf{v}_1$, $\mathbf{v}_3 = A\mathbf{v}_2 = A^2\mathbf{v}_1$, $\mathbf{v}_4 = A\mathbf{v}_3 = A^3\mathbf{v}_1$.
In general, $\mathbf{v}_n = A^{n-1}\mathbf{v}_1$.

Characteristic equation of the matrix A:

$$\begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0 \iff \lambda^2 - \lambda - 1 = 0.$$

Eigenvalues: $\lambda_1 = \frac{1+\sqrt{5}}{2}$, $\lambda_2 = \frac{1-\sqrt{5}}{2}$.

Let $\mathbf{w}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\mathbf{w}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ be eigenvectors of A associated with the eigenvalues λ_1 and λ_2. Then $\mathbf{w}_1, \mathbf{w}_2$ is a basis for $\mathbb{R}^2$.

In particular, $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = c_1\mathbf{w}_1 + c_2\mathbf{w}_2$ for some $c_1, c_2 \in \mathbb{R}$.
It follows that

$$\mathbf{v}_n = A^{n-1}\mathbf{v}_1 = A^{n-1}(c_1\mathbf{w}_1 + c_2\mathbf{w}_2)$$
$$= c_1A^{n-1}\mathbf{w}_1 + c_2A^{n-1}\mathbf{w}_2 = c_1\lambda_1^{n-1}\mathbf{w}_1 + c_2\lambda_2^{n-1}\mathbf{w}_2.$$

$$\mathbf{v}_n = c_1\lambda_1^{n-1}\mathbf{w}_1 + c_2\lambda_2^{n-1}\mathbf{w}_2$$

$$\Longrightarrow \quad f_n = c_1\lambda_1^{n-1}y_1 + c_2\lambda_2^{n-1}y_2.$$

Recall that $\lambda_1 = \frac{1+\sqrt{5}}{2}$, $\lambda_2 = \frac{1-\sqrt{5}}{2}$.

We have $\lambda_1 > 1$ and $-1 < \lambda_2 < 0$.

Therefore

$$\frac{f_{n+1}}{f_n} = \frac{c_1\lambda_1^n y_1 + c_2\lambda_2^n y_2}{c_1\lambda_1^{n-1}y_1 + c_2\lambda_2^{n-1}y_2}$$

$$= \lambda_1 \frac{c_1y_1 + c_2(\lambda_2/\lambda_1)^n y_2}{c_1y_1 + c_2(\lambda_2/\lambda_1)^{n-1}y_2} \to \lambda_1 \frac{c_1y_1}{c_1y_1} = \lambda_1$$

provided that $c_1y_1 \neq 0$.

Thus $\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \lambda_1 = \frac{1+\sqrt{5}}{2}$.