MATH 304

Linear Algebra

# Lecture 38:
# Rotations in space (continued).
# Orthogonal polynomials.

$U = \frac{1}{3}\begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{pmatrix}$ is an orthogonal matrix.

$\det U = 1 \implies U$ is a rotation matrix.

**Problem. (a)** Find the axis of the rotation.
**(b)** Find the angle of the rotation.

The axis is the set of points $\mathbf{x} \in \mathbb{R}^n$ such that $U\mathbf{x} = \mathbf{x} \iff (U - I)\mathbf{x} = \mathbf{0}$. To find the axis, we apply row reduction to the matrix

$$3(U - I) = 3U - 3I = \begin{pmatrix} -2 & -2 & 2 \\ 2 & -4 & -2 \\ 2 & 2 & -2 \end{pmatrix}.$$

$$\begin{pmatrix} -2 & -2 & 2 \\ 2 & -4 & -2 \\ 2 & 2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 \\ 2 & -4 & -2 \\ 2 & 2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 \\ 0 & -6 & 0 \\ 2 & 2 & -2 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & -1 \\ 0 & -6 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus $U\mathbf{x} = \mathbf{x} \iff \begin{cases} x - z = 0, \\ y = 0. \end{cases}$

The general solution is $x = t,\ y = 0,\ z = t,$ where $t \in \mathbb{R}$.

$\implies$ $\mathbf{d} = (1, 0, 1)$ is the direction of the axis.

$$U = \frac{1}{3}\begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{pmatrix}$$

Let $\phi$ be the angle of rotation. Then the eigenvalues of $U$ are 1, $e^{i\phi}$, and $e^{-i\phi}$. Therefore

$$\det(U - \lambda I) = (1-\lambda)(e^{i\phi} - \lambda)(e^{-i\phi} - \lambda).$$

Besides, $\det(U - \lambda I) = -\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3$, where $c_1 = \operatorname{tr} U$ (the sum of diagonal entries). It follows that

$$\operatorname{tr} U = 1 + e^{i\phi} + e^{-i\phi} = 1 + 2\cos\phi.$$

$$\operatorname{tr} U = 1/3 \implies \cos\phi = -1/3 \implies \phi \approx 109.47^{\mathrm{o}}$$

# Orthogonal polynomials

$\mathcal{P}$: the vector space of all polynomials with real coefficients: $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$.

Basis for $\mathcal{P}$: $1, x, x^2, \dots, x^n, \dots$

Suppose that $\mathcal{P}$ is endowed with an inner product.

*Definition.* **Orthogonal polynomials** (relative to the inner product) are polynomials $p_0, p_1, p_2, \dots$ such that $\deg p_n = n$ ($p_0$ is a nonzero constant) and $\langle p_n, p_m \rangle = 0$ for $n \neq m$.

Orthogonal polynomials can be obtained by applying the *Gram-Schmidt orthogonalization process* to the basis $1, x, x^2, \dots$:

$$p_0(x) = 1,$$

$$p_1(x) = x - \frac{\langle x, p_0 \rangle}{\langle p_0, p_0 \rangle} p_0(x),$$

$$p_2(x) = x^2 - \frac{\langle x^2, p_0 \rangle}{\langle p_0, p_0 \rangle} p_0(x) - \frac{\langle x^2, p_1 \rangle}{\langle p_1, p_1 \rangle} p_1(x),$$

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$$p_n(x) = x^n - \frac{\langle x^n, p_0 \rangle}{\langle p_0, p_0 \rangle} p_0(x) - \cdots - \frac{\langle x^n, p_{n-1} \rangle}{\langle p_{n-1}, p_{n-1} \rangle} p_{n-1}(x),$$

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Then $p_0, p_1, p_2, \dots$ are orthogonal polynomials.

**Theorem (a)** Orthogonal polynomials always exist.

**(b)** The orthogonal polynomial of a fixed degree is unique up to scaling.

**(c)** A polynomial $p \neq 0$ is an orthogonal polynomial if and only if $\langle p, q \rangle = 0$ for any polynomial $q$ with $\deg q < \deg p$.

**(d)** A polynomial $p \neq 0$ is an orthogonal polynomial if and only if $\langle p, x^k \rangle = 0$ for any $0 \leq k < \deg p$.

*Proof of statement (b):* Suppose that $P$ and $R$ are two orthogonal polynomials of the same degree $n$. Then $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ and $R(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0$, where $a_n, b_n \neq 0$. Consider a polynomial $Q(x) = b_n P(x) - a_n R(x)$. By construction, $\deg Q < n$. It follows from statement (c) that $\langle P, Q \rangle = \langle R, Q \rangle = 0$. Then

$$\langle Q, Q \rangle = \langle b_n P - a_n R, Q \rangle = b_n \langle P, Q \rangle - a_n \langle R, Q \rangle = 0,$$

which means that $Q = 0$. Thus $R(x) = (a_n^{-1} b_n)\, P(x)$.

*Example.* $\langle p, q\rangle = \int_{-1}^{1} p(x)q(x)\,dx.$

Note that $\langle x^n, x^m\rangle = 0$ if $m+n$ is odd.
Hence $p_{2k}(x)$ contains only even powers of $x$ while $p_{2k+1}(x)$ contains only odd powers of $x$.

$p_0(x) = 1,$
$p_1(x) = x,$

$$p_2(x) = x^2 - \frac{\langle x^2, 1\rangle}{\langle 1, 1\rangle} = x^2 - \frac{1}{3},$$

$$p_3(x) = x^3 - \frac{\langle x^3, x\rangle}{\langle x, x\rangle}x = x^3 - \frac{3}{5}x.$$

$p_0, p_1, p_2, \ldots$ are called the **Legendre polynomials**.

Instead of normalization, the orthogonal polynomials are subject to **standardization**.

The standardization for the Legendre polynomials is $P_n(1) = 1$. In particular, $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2 - 1)$, $P_3(x) = \frac{1}{2}(5x^3 - 3x)$.

**Problem.** Find $P_4(x)$.

Let $P_4(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$.
We know that $P_4(1) = 1$ and $\langle P_4, x^k \rangle = 0$ for $0 \le k \le 3$.

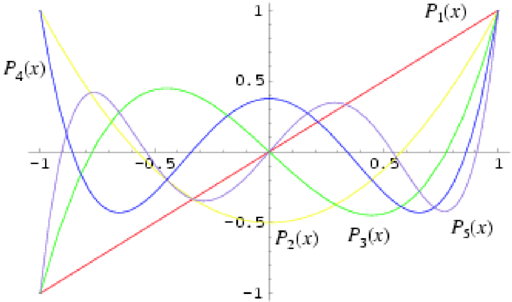
$P_4(1) = a_4 + a_3 + a_2 + a_1 + a_0$,
$\langle P_4, 1 \rangle = \frac{2}{5}a_4 + \frac{2}{3}a_2 + 2a_0$, $\langle P_4, x \rangle = \frac{2}{5}a_3 + \frac{2}{3}a_1$,
$\langle P_4, x^2 \rangle = \frac{2}{7}a_4 + \frac{2}{5}a_2 + \frac{2}{3}a_0$, $\langle P_4, x^3 \rangle = \frac{2}{7}a_3 + \frac{2}{5}a_1$.

$$\begin{cases} a_4 + a_3 + a_2 + a_1 + a_0 = 1 \\ \frac{2}{5}a_4 + \frac{2}{3}a_2 + 2a_0 = 0 \\ \frac{2}{5}a_3 + \frac{2}{3}a_1 = 0 \\ \frac{2}{7}a_4 + \frac{2}{5}a_2 + \frac{2}{3}a_0 = 0 \\ \frac{2}{7}a_3 + \frac{2}{5}a_1 = 0 \end{cases}$$

$$\begin{cases} \frac{2}{5}a_3 + \frac{2}{3}a_1 = 0 \\ \frac{2}{7}a_3 + \frac{2}{5}a_1 = 0 \end{cases} \implies a_1 = a_3 = 0$$

$$\begin{cases} a_4 + a_2 + a_0 = 1 \\ \frac{2}{5}a_4 + \frac{2}{3}a_2 + 2a_0 = 0 \\ \frac{2}{7}a_4 + \frac{2}{5}a_2 + \frac{2}{3}a_0 = 0 \end{cases} \iff \begin{cases} a_4 = \frac{35}{8} \\ a_2 = -\frac{30}{8} \\ a_0 = \frac{3}{8} \end{cases}$$

Thus $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$.




Legendre polynomials

**Problem.** Find a quadratic polynomial that is the best least squares fit to the function $f(x) = |x|$ on the interval $[-1, 1]$.

The best least squares fit is a polynomial $p(x)$ that minimizes the distance relative to the integral norm

$$\|f - p\| = \left( \int_{-1}^{1} |f(x) - p(x)|^2 \, dx \right)^{1/2}$$

over all polynomials of degree 2.

The norm $\|f - p\|$ is minimal if $p$ is the orthogonal projection of the function $f$ on the subspace $\mathcal{P}_3$ of polynomials of degree at most 2.

The Legendre polynomials $P_0, P_1, P_2$ form an orthogonal basis for $\mathcal{P}_3$. Therefore

$$p(x) = \frac{\langle f, P_0 \rangle}{\langle P_0, P_0 \rangle} P_0(x) + \frac{\langle f, P_1 \rangle}{\langle P_1, P_1 \rangle} P_1(x) + \frac{\langle f, P_2 \rangle}{\langle P_2, P_2 \rangle} P_2(x).$$

$$\langle f, P_0 \rangle = \int_{-1}^{1} |x| \, dx = 2 \int_0^1 x \, dx = 1,$$

$$\langle f, P_1 \rangle = \int_{-1}^{1} |x| \, x \, dx = 0,$$

$$\langle f, P_2 \rangle = \int_{-1}^{1} |x| \, \frac{3x^2 - 1}{2} \, dx = \int_0^1 x(3x^2 - 1) \, dx = \frac{1}{4},$$

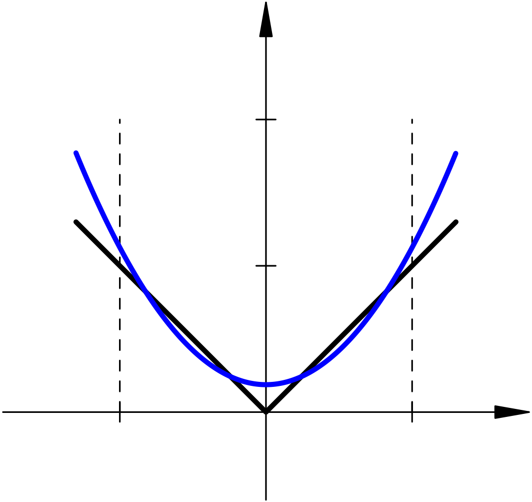
$$\langle P_0, P_0 \rangle = \int_{-1}^{1} dx = 2, \qquad \langle P_2, P_2 \rangle = \int_{-1}^{1} \left( \frac{3x^2 - 1}{2} \right)^2 dx = \frac{2}{5}.$$

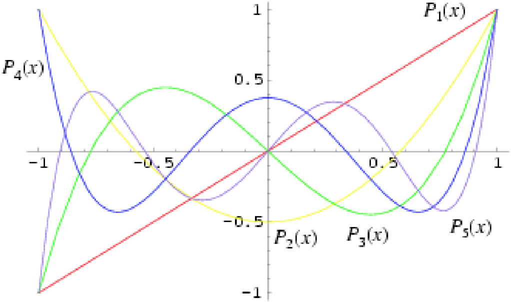
In general, $\langle P_n, P_n \rangle = \dfrac{2}{2n + 1}$.

**Problem.** Find a quadratic polynomial that is the best least squares fit to the function $f(x) = |x|$ on the interval $[-1, 1]$.

**Solution:** $$p(x) = \frac{1}{2}P_0(x) + \frac{5}{8}P_2(x)$$
$$= \frac{1}{2} + \frac{5}{16}(3x^2 - 1) = \frac{3}{16}(5x^2 + 1).$$

*Recurrent formula for the Legendre polynomials:*
$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x).$$
For example, $4P_4(x) = 7xP_3(x) - 3P_2(x)$.






Legendre polynomials

*Definition.* **Chebyshev polynomials** $T_0, T_1, T_2, \ldots$ are orthogonal polynomials relative to the inner product

$$\langle p, q \rangle = \int_{-1}^{1} \frac{p(x)q(x)}{\sqrt{1-x^2}}\, dx,$$

with the standardization $T_n(1) = 1$.

*Remark.* "T" is like in "Tschebyscheff".

Change of variable in the integral: $x = \cos\phi$.

$$\langle p, q \rangle = -\int_0^{\pi} \frac{p(\cos\phi)\, q(\cos\phi)}{\sqrt{1-\cos^2\phi}} \cos'\phi\, d\phi$$

$$= \int_0^{\pi} p(\cos\phi)\, q(\cos\phi)\, d\phi.$$

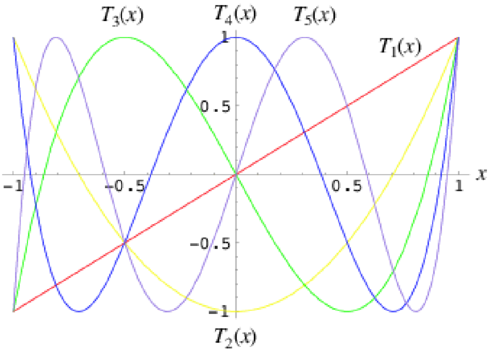
**Theorem.** $T_n(\cos\phi) = \cos n\phi$.

$$\langle T_n, T_m \rangle = \int_0^\pi T_n(\cos\phi)\, T_m(\cos\phi)\, d\phi$$
$$= \int_0^\pi \cos(n\phi)\cos(m\phi)\, d\phi = 0 \quad \text{if} \quad n \neq m.$$

Recurrent formula: $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$.

$T_0(x) = 1, \quad T_1(x) = x,$
$T_2(x) = 2x^2 - 1,$
$T_3(x) = 4x^3 - 3x,$
$T_4(x) = 8x^4 - 8x^2 + 1, \ \dots$

That is, $\cos 2\phi = 2\cos^2\phi - 1,$
$\cos 3\phi = 4\cos^3\phi - 3\cos\phi,$
$\cos 4\phi = 8\cos^4\phi - 8\cos^2\phi + 1, \ \dots$




Chebyshev polynomials