

MATH 304

Linear Algebra

**Lecture 3:**

**Row echelon form.**

**Gauss-Jordan reduction.**

## System of linear equations

$$\left\{ \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array} \right.$$

Here  $x_1, x_2, \dots, x_n$  are variables and  $a_{ij}, b_j$  are constants.

A *solution* of the system is a common solution of all equations in the system.

A system of linear equations can have **one** solution, **infinitely many** solutions, or **no** solution at all.

## Gaussian elimination

Solution of a system of linear equations splits into two parts: **(A)** elimination and **(B)** back substitution. Both parts can be done by applying a finite number of **elementary operations**.

*Elementary operations* for systems of linear equations:

- (1) to multiply an equation by a nonzero scalar;
- (2) to add an equation multiplied by a scalar to another equation;
- (3) to interchange two equations.

System of linear equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

Coefficient matrix and column vector of the right-hand sides:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

System of linear equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

*Augmented matrix:*

$$\left( \begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array} \right)$$

Since the elementary operations preserve the standard form of linear equations, we can trace the solution process by looking on the **augmented matrix**.

Elementary operations for systems of linear equations correspond to **elementary row operations** for augmented matrices:

- (1) to multiply a row by a nonzero scalar;
- (2) to add the  $i$ th row multiplied by some  $r \in \mathbb{R}$  to the  $j$ th row;
- (3) to interchange two rows.

*Remark.* Rows are added and multiplied by scalars as vectors (namely, row vectors).



## Row echelon form

General augmented matrix in row echelon form:

$$\left( \begin{array}{cccccccccccc|c} \boxed{\phantom{0}} & * & * & * & * & * & * & * & * & * & * & * & * \\ \boxed{\phantom{0}} & \circledast & \circledast & * & * & * & * & * & * & * & * & * & * \\ & \boxed{\phantom{0}} & \circledast & * & * & * & * & * & * & * & * & * & * \\ & & \boxed{\phantom{0}} & \circledast & * & * & * & * & * & * & * & * & * \\ & & & \boxed{\phantom{0}} & * & * & * & * & * & * & * & * & * \\ & & & & \boxed{\phantom{0}} & * & * & * & * & * & * & * & * \\ & & & & & \boxed{\phantom{0}} & \circledast & \circledast & * & * & * & * & * \\ & & & & & & & & & & & & * \end{array} \right)$$

- leading entries are boxed (all equal to 1);
- all the entries below the staircase line are zero;
- each step of the staircase has height 1;
- each circle marks a column without a leading entry that corresponds to a free variable.



**Strict triangular form** is a particular case of row echelon form that can occur for systems of  $n$  equations in  $n$  variables:

$$\left( \begin{array}{cccccccc|c} \square & * & * & * & * & * & * & * & * \\ & \square & * & * & * & * & * & * & * \\ & & \square & * & * & * & * & * & * \\ & & & \square & * & * & * & * & * \\ & & & & \square & * & * & * & * \\ & & & & & \square & * & * & * \\ & & & & & & \square & * & * \\ & & & & & & & \square & * \end{array} \right)$$

- no zero rows;
- no free variables.

## Consistency check

The original system of linear equations is **consistent** if there is no leading entry in the rightmost column of the augmented matrix in row echelon form.

The diagram shows an augmented matrix in row echelon form. The matrix is enclosed in large parentheses. A vertical line separates the coefficient matrix from the augmented column. A blue staircase line indicates the leading entries in each row. The rightmost column contains asterisks in every row, indicating a leading entry in that column. Red circles highlight the asterisks in the second, third, and sixth rows of the augmented column.

$$\left( \begin{array}{cccccccccccc|c} \square & * & * & * & * & * & * & * & * & * & * & * \\ \square & * & * & * & * & * & * & * & * & * & * & * \\ & \square & * & * & * & * & * & * & * & * & * & * \\ & & \square & * & * & * & * & * & * & * & * & * \\ & & & \square & * & * & * & * & * & * & * & * \\ & & & & \square & * & * & * & * & * & * & * \\ & & & & & \square & * & * & * & * & * & * \\ & & & & & & \square & * & * & * & * & * \\ & & & & & & & \square & * & * & * & * \\ & & & & & & & & \square & * & * & * \\ & & & & & & & & & \square & * & * \\ & & & & & & & & & & \square & * \end{array} \right)$$

Augmented matrix of an inconsistent system

The goal of the **Gauss-Jordan reduction** is to convert the augmented matrix into **reduced row echelon form**:

$$\left( \begin{array}{cccc|ccc} \boxed{1} & * & * & * & * & * & * \\ & \boxed{1} & \circledast & \circledast & * & * & * \\ & & & \boxed{1} & \circledast & * & * \\ & & & & \boxed{1} & * & * \\ & & & & & \boxed{1} & \circledast & \circledast \\ & & & & & & & * \end{array} \right)$$

- all entries below the staircase line are zero;
- each boxed entry is 1, the other entries in its column are zero;
- each circle corresponds to a free variable.

### Example.

$$\begin{cases} x - y & = 2 \\ 2x - y - z & = 3 \\ x + y + z & = 6 \end{cases} \quad \left( \begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 2 & -1 & -1 & 3 \\ 1 & 1 & 1 & 6 \end{array} \right)$$

Row echelon form (also strict triangular):

$$\begin{cases} x - y & = 2 \\ y - z & = -1 \\ z & = 2 \end{cases} \quad \left( \begin{array}{ccc|c} \boxed{1} & -1 & 0 & 2 \\ 0 & \boxed{1} & -1 & -1 \\ 0 & 0 & \boxed{1} & 2 \end{array} \right)$$

Reduced row echelon form:

$$\begin{cases} x & = 3 \\ y & = 1 \\ z & = 2 \end{cases} \quad \left( \begin{array}{ccc|c} \boxed{1} & 0 & 0 & 3 \\ 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & \boxed{1} & 2 \end{array} \right)$$

## Another example.

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ -x + 4y - 3z = 14 \end{cases} \quad \left( \begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & 1 & -1 & 3 \\ -1 & 4 & -3 & 14 \end{array} \right)$$

Row echelon form:

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ 0 = 0 \end{cases} \quad \left( \begin{array}{ccc|c} \boxed{1} & 1 & -2 & 1 \\ 0 & \boxed{1} & -1 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

Reduced row echelon form:

$$\begin{cases} x - z = -2 \\ y - z = 3 \\ 0 = 0 \end{cases} \quad \left( \begin{array}{ccc|c} \boxed{1} & 0 & -1 & -2 \\ 0 & \boxed{1} & -1 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

## Yet another example.

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ -x + 4y - 3z = 1 \end{cases} \quad \left( \begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & 1 & -1 & 3 \\ -1 & 4 & -3 & 1 \end{array} \right)$$

Row echelon form:

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ 0 = 1 \end{cases} \quad \left( \begin{array}{ccc|c} \boxed{1} & 1 & -2 & 1 \\ 0 & \boxed{1} & -1 & 3 \\ 0 & 0 & 0 & \boxed{1} \end{array} \right)$$

Reduced row echelon form:

$$\begin{cases} x - z = 0 \\ y - z = 0 \\ 0 = 1 \end{cases} \quad \left( \begin{array}{ccc|c} \boxed{1} & 0 & -1 & 0 \\ 0 & \boxed{1} & -1 & 0 \\ 0 & 0 & 0 & \boxed{1} \end{array} \right)$$

## How to solve a system of linear equations

- Order the variables.
- Write down the augmented matrix of the system.
- Convert the matrix to **row echelon form**.
- Check for consistency.
- Convert the matrix to **reduced row echelon form**.
- Write down the system corresponding to the reduced row echelon form.
- Determine leading and free variables.
- Rewrite the system so that the leading variables are on the left while everything else is on the right.
- Assign parameters to the free variables and write down the general solution in parametric form.

**New example.** 
$$\begin{cases} x_2 + 2x_3 + 3x_4 = 6 \\ x_1 + 2x_2 + 3x_3 + 4x_4 = 10 \end{cases}$$

Variables:  $x_1, x_2, x_3, x_4$ .

Augmented matrix: 
$$\left( \begin{array}{cccc|c} 0 & 1 & 2 & 3 & 6 \\ 1 & 2 & 3 & 4 & 10 \end{array} \right)$$

To get it into row echelon form, we exchange the two rows:

$$\left( \begin{array}{cccc|c} 1 & 2 & 3 & 4 & 10 \\ 0 & 1 & 2 & 3 & 6 \end{array} \right)$$

Consistency check is passed. To convert into reduced row echelon form, add  $-2$  times the 2nd row to the 1st row:

$$\left( \begin{array}{cccc|c} \boxed{1} & 0 & -1 & -2 & -2 \\ 0 & \boxed{1} & 2 & 3 & 6 \end{array} \right)$$

The leading variables are  $x_1$  and  $x_2$ ; hence  $x_3$  and  $x_4$  are free variables.



Back to the system:

$$\begin{cases} x_1 - x_3 - 2x_4 = -2 \\ x_2 + 2x_3 + 3x_4 = 6 \end{cases} \iff \begin{cases} x_1 = x_3 + 2x_4 - 2 \\ x_2 = -2x_3 - 3x_4 + 6 \end{cases}$$

General solution:

$$\begin{cases} x_1 = t + 2s - 2 \\ x_2 = -2t - 3s + 6 \\ x_3 = t \\ x_4 = s \end{cases} \quad (t, s \in \mathbb{R})$$

In vector form,  $(x_1, x_2, x_3, x_4) =$   
 $= (-2, 6, 0, 0) + t(1, -2, 1, 0) + s(2, -3, 0, 1).$

### Example with a parameter.

$$\begin{cases} y + 3z = 0 \\ x + y - 2z = 0 \\ x + 2y + az = 0 \end{cases} \quad (a \in \mathbb{R})$$

The system is **homogeneous** (all right-hand sides are zeros). Therefore it is consistent ( $x = y = z = 0$  is a solution).

Augmented matrix: 
$$\left( \begin{array}{ccc|c} 0 & 1 & 3 & 0 \\ 1 & 1 & -2 & 0 \\ 1 & 2 & a & 0 \end{array} \right)$$

Since the 1st row cannot serve as a pivotal one, we interchange it with the 2nd row:

$$\left( \begin{array}{ccc|c} 0 & 1 & 3 & 0 \\ 1 & 1 & -2 & 0 \\ 1 & 2 & a & 0 \end{array} \right) \rightarrow \left( \begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 1 & 2 & a & 0 \end{array} \right)$$

Now we can start the elimination.

First subtract the 1st row from the 3rd row:

$$\left( \begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 1 & 2 & a & 0 \end{array} \right) \rightarrow \left( \begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & a+2 & 0 \end{array} \right)$$

The 2nd row is our new pivotal row.

Subtract the 2nd row from the 3rd row:

$$\left( \begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & a+2 & 0 \end{array} \right) \rightarrow \left( \begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & a-1 & 0 \end{array} \right)$$

At this point row reduction splits into two cases.

**Case 1:**  $a \neq 1$ . In this case, multiply the 3rd row by  $(a - 1)^{-1}$ :

$$\left( \begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & a-1 & 0 \end{array} \right) \rightarrow \left( \begin{array}{ccc|c} \boxed{1} & 1 & -2 & 0 \\ 0 & \boxed{1} & 3 & 0 \\ 0 & 0 & \boxed{1} & 0 \end{array} \right)$$

*The matrix is converted into row echelon form.*

*We proceed towards reduced row echelon form.*

Subtract 3 times the 3rd row from the 2nd row:

$$\left( \begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right) \rightarrow \left( \begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right)$$

Add 2 times the 3rd row to the 1st row:

$$\left( \begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right) \rightarrow \left( \begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right)$$

Finally, subtract the 2nd row from the 1st row:

$$\left( \begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right) \rightarrow \left( \begin{array}{ccc|c} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \end{array} \right)$$

Thus  $x = y = z = 0$  is the only solution.

**Case 2:**  $a = 1$ . In this case, the matrix is already in row echelon form:

$$\left( \begin{array}{ccc|c} \boxed{1} & 1 & -2 & 0 \\ 0 & \boxed{1} & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

To get reduced row echelon form, subtract the 2nd row from the 1st row:

$$\left( \begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \rightarrow \left( \begin{array}{ccc|c} \boxed{1} & 0 & -5 & 0 \\ 0 & \boxed{1} & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$z$  is a free variable.

$$\begin{cases} x - 5z = 0 \\ y + 3z = 0 \end{cases} \iff \begin{cases} x = 5z \\ y = -3z \end{cases}$$

## System of linear equations:

$$\begin{cases} y + 3z = 0 \\ x + y - 2z = 0 \\ x + 2y + az = 0 \end{cases}$$

**Solution:** If  $a \neq 1$  then  $(x, y, z) = (0, 0, 0)$ ;  
if  $a = 1$  then  $(x, y, z) = (5t, -3t, t)$ ,  $t \in \mathbb{R}$ .