

MATH 304  
Linear Algebra

**Lecture 8:**  
**Elementary matrices.**  
**Transpose of a matrix.**  
**Determinants.**

## General results on inverse matrices

**Theorem 1** Given a square matrix  $A$ , the following are equivalent:

- (i)  $A$  is invertible;
- (ii)  $\mathbf{x} = \mathbf{0}$  is the only solution of the matrix equation  $A\mathbf{x} = \mathbf{0}$ ;
- (iii) the row echelon form of  $A$  has no zero rows;
- (iv) the reduced row echelon form of  $A$  is the identity matrix.

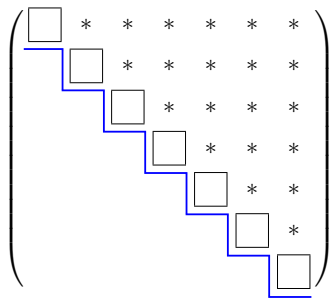
**Theorem 2** Suppose that a sequence of elementary row operations converts a matrix  $A$  into the identity matrix.

Then the same sequence of operations converts the identity matrix into the inverse matrix  $A^{-1}$ .

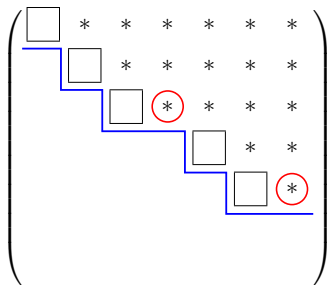
**Theorem 3** For any  $n \times n$  matrices  $A$  and  $B$ ,

$$BA = I \iff AB = I.$$

*Row echelon form of a square matrix:*



invertible case



noninvertible case

## Why does it work?

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 \\ 2b_1 & 2b_2 & 2b_3 \\ c_1 & c_2 & c_3 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1+3a_1 & b_2+3a_2 & b_3+3a_3 \\ c_1 & c_2 & c_3 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{pmatrix}.$$

**Proposition** Any elementary row operation can be simulated as left multiplication by a certain matrix.



## Elementary matrices

$$E = \begin{pmatrix} 1 & & & & & & & \\ \vdots & \ddots & & & & & & \\ 0 & \cdots & 1 & & & & & \\ \vdots & & \vdots & \ddots & & & & \\ 0 & \cdots & r & \cdots & 1 & & & \\ \vdots & & \vdots & & \vdots & \ddots & & \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 & \end{pmatrix} \begin{array}{l} \text{row \#}i \\ \\ \text{row \#}j \end{array}$$

To obtain the matrix  $EA$  from  $A$ , add  $r$  times the  $i$ th row to the  $j$ th row. To obtain the matrix  $AE$  from  $A$ , add  $r$  times the  $j$ th column to the  $i$ th column.



## Why does it work?

Assume that a square matrix  $A$  can be converted to the identity matrix by a sequence of elementary row operations. Then

$$E_k E_{k-1} \dots E_2 E_1 A = I,$$

where  $E_1, E_2, \dots, E_k$  are elementary matrices simulating those operations.

Applying the same sequence of operations to the identity matrix, we obtain the matrix

$$B = E_k E_{k-1} \dots E_2 E_1 I = E_k E_{k-1} \dots E_2 E_1.$$

Thus  $BA = I$ , which implies that  $B = A^{-1}$ .



## Transpose of a matrix

*Definition.* Given a matrix  $A$ , the **transpose** of  $A$ , denoted  $A^T$ , is the matrix whose rows are columns of  $A$  (and whose columns are rows of  $A$ ). That is, if  $A = (a_{ij})$  then  $A^T = (b_{ij})$ , where  $b_{ij} = a_{ji}$ .

*Examples.* 
$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix},$$

$$\begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}^T = (7, 8, 9), \quad \begin{pmatrix} 4 & 7 \\ 7 & 0 \end{pmatrix}^T = \begin{pmatrix} 4 & 7 \\ 7 & 0 \end{pmatrix}.$$

## Properties of transposes:

- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$
- $(rA)^T = rA^T$
- $(AB)^T = B^T A^T$
- $(A_1 A_2 \dots A_k)^T = A_k^T \dots A_2^T A_1^T$
- $(A^{-1})^T = (A^T)^{-1}$

*Definition.* A square matrix  $A$  is said to be **symmetric** if  $A^T = A$ .

For example, any diagonal matrix is symmetric.

**Proposition** For any square matrix  $A$  the matrices  $B = AA^T$  and  $C = A + A^T$  are symmetric.

*Proof:*

$$B^T = (AA^T)^T = (A^T)^T A^T = AA^T = B,$$

$$C^T = (A + A^T)^T = A^T + (A^T)^T = A^T + A = C.$$

## Determinants

**Determinant** is a scalar assigned to each square matrix.

*Notation.* The determinant of a matrix

$A = (a_{ij})_{1 \leq i, j \leq n}$  is denoted  $\det A$  or

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}.$$

**Principal property:**  $\det A \neq 0$  if and only if a system of linear equations with the coefficient matrix  $A$  has a unique solution. Equivalently,  $\det A \neq 0$  if and only if the matrix  $A$  is invertible.

## Definition in low dimensions

*Definition.*  $\det(a) = a$ ,  $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ ,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - \\ - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.$$

$$+ : \begin{pmatrix} \boxed{*} & * & * \\ * & \boxed{*} & * \\ * & * & \boxed{*} \end{pmatrix}, \begin{pmatrix} * & \boxed{*} & * \\ * & * & \boxed{*} \\ \boxed{*} & * & * \end{pmatrix}, \begin{pmatrix} * & * & \boxed{*} \\ \boxed{*} & * & * \\ * & \boxed{*} & * \end{pmatrix}.$$

$$- : \begin{pmatrix} * & * & \boxed{*} \\ * & \boxed{*} & * \\ \boxed{*} & * & * \end{pmatrix}, \begin{pmatrix} * & \boxed{*} & * \\ \boxed{*} & * & * \\ * & * & \boxed{*} \end{pmatrix}, \begin{pmatrix} \boxed{*} & * & * \\ * & * & \boxed{*} \\ * & \boxed{*} & * \end{pmatrix}.$$

## Examples: $2 \times 2$ matrices

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1, \quad \begin{vmatrix} 3 & 0 \\ 0 & -4 \end{vmatrix} = -12,$$

$$\begin{vmatrix} -2 & 5 \\ 0 & 3 \end{vmatrix} = -6, \quad \begin{vmatrix} 7 & 0 \\ 5 & 2 \end{vmatrix} = 14,$$

$$\begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} = 1, \quad \begin{vmatrix} 0 & 0 \\ 4 & 1 \end{vmatrix} = 0,$$

$$\begin{vmatrix} -1 & 3 \\ -1 & 3 \end{vmatrix} = 0, \quad \begin{vmatrix} 2 & 1 \\ 8 & 4 \end{vmatrix} = 0.$$

## Examples: $3 \times 3$ matrices

$$\begin{vmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{vmatrix} = 3 \cdot 0 \cdot 0 + (-2) \cdot 1 \cdot (-2) + 0 \cdot 1 \cdot 3 - \\ - 0 \cdot 0 \cdot (-2) - (-2) \cdot 1 \cdot 0 - 3 \cdot 1 \cdot 3 = 4 - 9 = -5,$$

$$\begin{vmatrix} 1 & 4 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{vmatrix} = 1 \cdot 2 \cdot 3 + 4 \cdot 5 \cdot 0 + 6 \cdot 0 \cdot 0 - \\ - 6 \cdot 2 \cdot 0 - 4 \cdot 0 \cdot 3 - 1 \cdot 5 \cdot 0 = 1 \cdot 2 \cdot 3 = 6.$$