MATH 304

Linear Algebra

Lecture 12:
Subspaces of vector spaces.

Vector space

A *vector space* is a set V equipped with two operations, **addition**

$$V \times V \ni (\mathbf{x}, \mathbf{y}) \mapsto \mathbf{x} + \mathbf{y} \in V$$

and **scalar multiplication**

$$\mathbb{R} \times V \ni (r, \mathbf{x}) \mapsto r\mathbf{x} \in V,$$

that have the following properties:

Properties of addition and scalar multiplication

A1. $\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}$ for all $\mathbf{a},\mathbf{b}\in V$.

A2. $(\mathbf{a}+\mathbf{b})+\mathbf{c}=\mathbf{a}+(\mathbf{b}+\mathbf{c})$ for all $\mathbf{a},\mathbf{b},\mathbf{c}\in V$.

A3. There exists an element of V, called the *zero vector* and denoted $\mathbf{0}$, such that $\mathbf{a}+\mathbf{0}=\mathbf{0}+\mathbf{a}=\mathbf{a}$ for all $\mathbf{a}\in V$.

A4. For any $\mathbf{a}\in V$ there exists an element of V, denoted $-\mathbf{a}$, such that $\mathbf{a}+(-\mathbf{a})=(-\mathbf{a})+\mathbf{a}=\mathbf{0}$.

A5. $r(\mathbf{a}+\mathbf{b})=r\mathbf{a}+r\mathbf{b}$ for all $r\in\mathbb{R}$ and $\mathbf{a},\mathbf{b}\in V$.

A6. $(r+s)\mathbf{a}=r\mathbf{a}+s\mathbf{a}$ for all $r,s\in\mathbb{R}$ and $\mathbf{a}\in V$.

A7. $(rs)\mathbf{a}=r(s\mathbf{a})$ for all $r,s\in\mathbb{R}$ and $\mathbf{a}\in V$.

A8. $1\mathbf{a}=\mathbf{a}$ for all $\mathbf{a}\in V$.

- Associativity of addition implies that a multiple sum $\mathbf{u}_1+\mathbf{u}_2+\cdots+\mathbf{u}_k$ is well defined for any $\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_k \in V$.

- **Subtraction** in V is defined as usual: $\mathbf{a}-\mathbf{b}=\mathbf{a}+(-\mathbf{b})$.

- Addition and scalar multiplication are called **linear operations**.

Given $\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_k \in V$ and $r_1,r_2,\ldots,r_k \in \mathbb{R}$,

$$\boxed{r_1\mathbf{u}_1+r_2\mathbf{u}_2+\cdots+r_k\mathbf{u}_k}$$

is called a **linear combination** of $\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_k$.

Additional properties of vector spaces

- The zero vector is unique.
- For any $\mathbf{a} \in V$, the negative $-\mathbf{a}$ is unique.
- $\mathbf{a} + \mathbf{b} = \mathbf{c} \iff \mathbf{a} = \mathbf{c} - \mathbf{b}$ for all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in V$.
- $\mathbf{a} + \mathbf{c} = \mathbf{b} + \mathbf{c} \iff \mathbf{a} = \mathbf{b}$ for all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in V$.
- $0\mathbf{a} = \mathbf{0}$ for any $\mathbf{a} \in V$.
- $(-1)\mathbf{a} = -\mathbf{a}$ for any $\mathbf{a} \in V$.

Examples of vector spaces

- $\mathbb{R}^n$: n-dimensional coordinate vectors
- $\mathcal{M}_{m,n}(\mathbb{R})$: $m\times n$ matrices with real entries
- $\mathbb{R}^\infty$: infinite sequences $(x_1, x_2, \dots)$, $x_i \in \mathbb{R}$
- $\{\mathbf{0}\}$: the trivial vector space
- $F(\mathbb{R})$: the set of all functions $f : \mathbb{R} \to \mathbb{R}$
- $C(\mathbb{R})$: all continuous functions $f : \mathbb{R} \to \mathbb{R}$
- $C^1(\mathbb{R})$: all continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}$
- $C^\infty(\mathbb{R})$: all smooth functions $f : \mathbb{R} \to \mathbb{R}$
- $\mathcal{P}$: all polynomials $p(x) = a_0 + a_1x + \cdots + a_nx^n$

Subspaces of vector spaces

Definition. A vector space V_0 is a **subspace** of a vector space V if $V_0 \subset V$ and the linear operations on V_0 agree with the linear operations on V.

Examples.

- $F(\mathbb{R})$: all functions $f : \mathbb{R} \to \mathbb{R}$
- $C(\mathbb{R})$: all continuous functions $f : \mathbb{R} \to \mathbb{R}$

$C(\mathbb{R})$ is a subspace of $F(\mathbb{R})$.

- $\mathcal{P}$: polynomials $p(x) = a_0 + a_1 x + \cdots + a_k x^k$
- $\mathcal{P}_n$: polynomials of degree less than n

$\mathcal{P}_n$ is a subspace of $\mathcal{P}$.

Subspaces of vector spaces

Counterexamples.

- $\mathbb{R}^n$: n-dimensional coordinate vectors
- $\mathbb{Q}^n$: vectors with rational coordinates

$\mathbb{Q}^n$ is not a subspace of $\mathbb{R}^n$.

$\sqrt{2}(1,1,\dots,1) \notin \mathbb{Q}^n \implies \mathbb{Q}^n$ is not a vector space (scaling is not well defined).

- $\mathcal{P}$: polynomials $p(x) = a_0 + a_1 x + \cdots + a_n x^n$
- P_n^*: polynomials of degree n $(n > 0)$

P_n^* is not a subspace of $\mathcal{P}$.

$-x^n + (x^n + 1) = 1 \notin P_n^* \implies P_n^*$ is not a vector space (addition is not well defined).

If S is a subset of a vector space V then S inherits from V addition and scalar multiplication. However S need not be closed under these operations.

Proposition A subset S of a vector space V is a subspace of V if and only if S is **nonempty** and **closed under linear operations**, i.e.,

$$\mathbf{x}, \mathbf{y} \in S \implies \mathbf{x} + \mathbf{y} \in S,$$

$$\mathbf{x} \in S \implies r\mathbf{x} \in S \text{ for all } r \in \mathbb{R}.$$

Proof: “only if” is obvious.
“if”: properties like associative, commutative, or distributive law hold for S because they hold for V. We only need to verify properties A3 and A4. Take any $\mathbf{x} \in S$ (note that S is nonempty). Then $\mathbf{0} = 0\mathbf{x} \in S$. Also, $-\mathbf{x} = (-1)\mathbf{x} \in S$. Thus $\mathbf{0}$ and $-\mathbf{x}$ in S are the same as in V.

Example. $V = \mathbb{R}^2$.

- The line $x - y = 0$ is a subspace of $\mathbb{R}^2$.

The line consists of all vectors of the form (t, t), $t \in \mathbb{R}$.

$(t, t) + (s, s) = (t + s, t + s) \implies$ closed under addition
$r(t, t) = (rt, rt) \implies$ closed under scaling

- The parabola $y = x^2$ is not a subspace of $\mathbb{R}^2$.

It is enough to find one explicit counterexample.

Counterexample 1: $(1, 1) + (-1, 1) = (0, 2)$.

$(1, 1)$ and $(-1, 1)$ lie on the parabola while $(0, 2)$ does not
$\implies$ not closed under addition

Counterexample 2: $2(1, 1) = (2, 2)$.

$(1, 1)$ lies on the parabola while $(2, 2)$ does not
$\implies$ not closed under scaling

Example. $V = \mathbb{R}^3$.

- The plane $z = 0$ is a subspace of $\mathbb{R}^3$.
- The plane $z = 1$ is not a subspace of $\mathbb{R}^3$.
- The line $t(1, 1, 0)$, $t \in \mathbb{R}$ is a subspace of $\mathbb{R}^3$ and a subspace of the plane $z = 0$.
- The line $(1, 1, 1) + t(1, -1, 0)$, $t \in \mathbb{R}$ is not a subspace of $\mathbb{R}^3$ as it lies in the plane $x + y + z = 3$, which does not contain $\mathbf{0}$.
- In general, a straight line or a plane in $\mathbb{R}^3$ is a subspace if and only if it passes through the origin.

System of linear equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

Any solution $(x_1, x_2, \ldots, x_n)$ is an element of $\mathbb{R}^n$.

Theorem The solution set of the system is a subspace of $\mathbb{R}^n$ if and only if all $b_i = 0$.

Theorem The solution set of a system of linear equations in n variables is a subspace of $\mathbb{R}^n$ if and only if all equations are homogeneous.

Proof: "only if": the zero vector $\mathbf{0} = (0, 0, \dots, 0)$ is a solution only if all equations are homogeneous.

"if": a system of homogeneous linear equations is equivalent to a matrix equation $A\mathbf{x} = \mathbf{0}$, where A is the coefficient matrix of the system and all vectors are regarded as column vectors.

$A\mathbf{0} = \mathbf{0} \implies \mathbf{0}$ is a solution $\implies$ solution set is not empty.

If $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{y} = \mathbf{0}$ then $A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \mathbf{0}$
$\implies$ solution set is closed under addition.

If $A\mathbf{x} = \mathbf{0}$ then $A(r\mathbf{x}) = r(A\mathbf{x}) = \mathbf{0}$
$\implies$ solution set is closed under scaling.

Examples of subspaces of $\mathcal{M}_{2,2}(\mathbb{R})$*:* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

- diagonal matrices: $b = c = 0$
- upper triangular matrices: $c = 0$
- lower triangular matrices: $b = 0$
- symmetric matrices ($A^T = A$): $b = c$
- anti-symmetric (or skew-symmetric) matrices ($A^T = -A$): $a = d = 0,\ c = -b$
- matrices with zero trace: $a + d = 0$ (trace = the sum of diagonal entries)
- matrices with zero determinant, $ad - bc = 0$, **do not** form a subspace: $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Let V be a vector space and $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n \in V$. Consider the set L of all linear combinations $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_n\mathbf{v}_n$, where $r_1, r_2, \dots, r_n \in \mathbb{R}$.

Theorem L is a subspace of V.

Proof: First of all, L is not empty. For example, $\mathbf{0} = 0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_n$ belongs to L.

The set L is closed under addition since

$$(r_1\mathbf{v}_1+r_2\mathbf{v}_2+\cdots+r_n\mathbf{v}_n) + (s_1\mathbf{v}_1+s_2\mathbf{v}_2+\cdots+s_n\mathbf{v}_n) =$$
$$= (r_1+s_1)\mathbf{v}_1 + (r_2+s_2)\mathbf{v}_2 + \cdots + (r_n+s_n)\mathbf{v}_n.$$

The set L is closed under scalar multiplication since

$$t(r_1\mathbf{v}_1+r_2\mathbf{v}_2+\cdots+r_n\mathbf{v}_n) = (tr_1)\mathbf{v}_1+(tr_2)\mathbf{v}_2+\cdots+(tr_n)\mathbf{v}_n.$$