MATH 304

Linear Algebra

Lecture 13:
Span. Spanning set.

Subspaces of vector spaces

Definition. A vector space V_0 is a **subspace** of a vector space V if $V_0 \subset V$ and the linear operations on V_0 agree with the linear operations on V.

Proposition A subset S of a vector space V is a subspace of V if and only if S is **nonempty** and **closed under linear operations**, i.e.,

$$\mathbf{x}, \mathbf{y} \in S \implies \mathbf{x} + \mathbf{y} \in S,$$

$$\mathbf{x} \in S \implies r\mathbf{x} \in S \text{ for all } r \in \mathbb{R}.$$

Remarks. The zero vector in a subspace is the same as the zero vector in V. Also, the subtraction in a subspace agrees with that in V.

Examples of subspaces

- $F(\mathbb{R})$: all functions $f : \mathbb{R} \to \mathbb{R}$
- $C(\mathbb{R})$: all continuous functions $f : \mathbb{R} \to \mathbb{R}$

$C(\mathbb{R})$ is a subspace of $F(\mathbb{R})$.

- $\mathcal{P}$: polynomials $p(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}$
- $\mathcal{P}_n$: polynomials of degree less than n

$\mathcal{P}_n$ is a subspace of $\mathcal{P}$.

- Any vector space V
- $\{\mathbf{0}\}$, where $\mathbf{0}$ is the zero vector in V

The trivial space $\{\mathbf{0}\}$ is a subspace of V.

System of linear equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

Any solution $(x_1, x_2, \ldots, x_n)$ is an element of $\mathbb{R}^n$.

Theorem The solution set of the system is a subspace of $\mathbb{R}^n$ if and only if all $b_i = 0$.

Examples of subspaces of $\mathcal{M}_{2,2}(\mathbb{R})$*:* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

- diagonal matrices: $b = c = 0$
- upper triangular matrices: $c = 0$
- lower triangular matrices: $b = 0$
- symmetric matrices ($A^T = A$): $b = c$
- anti-symmetric matrices ($A^T = -A$):

$a = d = 0$ and $c = -b$

- matrices with zero trace: $a + d = 0$

(trace = the sum of diagonal entries)

Let V be a vector space and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in V$. Consider the set L of all linear combinations $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_n\mathbf{v}_n$, where $r_1, r_2, \ldots, r_n \in \mathbb{R}$.

Theorem L is a subspace of V.

Proof: First of all, L is not empty. For example, $\mathbf{0} = 0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_n$ belongs to L.

The set L is closed under addition since

$$(r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_n\mathbf{v}_n) + (s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + \cdots + s_n\mathbf{v}_n) =$$
$$= (r_1 + s_1)\mathbf{v}_1 + (r_2 + s_2)\mathbf{v}_2 + \cdots + (r_n + s_n)\mathbf{v}_n.$$

The set L is closed under scalar multiplication since

$$t(r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_n\mathbf{v}_n) = (tr_1)\mathbf{v}_1 + (tr_2)\mathbf{v}_2 + \cdots + (tr_n)\mathbf{v}_n.$$

Span: implicit definition

Let S be a subset of a vector space V.

Definition. The **span** of the set S, denoted $\mathrm{Span}(S)$, is the smallest subspace of V that contains S. That is,

- $\mathrm{Span}(S)$ is a subspace of V;
- for any subspace $W \subset V$ one has
$$S \subset W \implies \mathrm{Span}(S) \subset W.$$

Remark. The span of any set $S \subset V$ is well defined (it is the intersection of all subspaces of V that contain S).

Span: effective description

Let S be a subset of a vector space V.

- If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ then $\mathrm{Span}(S)$ is the set of all linear combinations $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_n\mathbf{v}_n$, where $r_1, r_2, \dots, r_n \in \mathbb{R}$.

- If S is an infinite set then $\mathrm{Span}(S)$ is the set of all linear combinations $r_1\mathbf{u}_1 + r_2\mathbf{u}_2 + \cdots + r_k\mathbf{u}_k$, where $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k \in S$ and $r_1, r_2, \dots, r_k \in \mathbb{R}$ $(k \geq 1)$.

- If S is the empty set then $\mathrm{Span}(S) = \{\mathbf{0}\}$.

Examples of subspaces of $\mathcal{M}_{2,2}(\mathbb{R})$:

- The span of $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ consists of all matrices of the form

$$a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

This is the subspace of diagonal matrices.

- The span of $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ consists of all matrices of the form

$$a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + c\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a & c \\ c & b \end{pmatrix}.$$

This is the subspace of symmetric matrices.

Examples of subspaces of $\mathcal{M}_{2,2}(\mathbb{R})$:

- The span of $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the subspace of anti-symmetric matrices.

- The span of $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, and $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is the subspace of upper triangular matrices.

- The span of $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ is the entire space $\mathcal{M}_{2,2}(\mathbb{R})$.

Spanning set

Definition. A subset S of a vector space V is called a **spanning set** for V if $\mathrm{Span}(S) = V$.

Examples.

- Vectors $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, and $\mathbf{e}_3 = (0, 0, 1)$ form a spanning set for $\mathbb{R}^3$ as

$$(x, y, z) = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3.$$

- Matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

form a spanning set for $\mathcal{M}_{2,2}(\mathbb{R})$ as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Problem Let $\mathbf{v}_1 = (1, 2, 0)$, $\mathbf{v}_2 = (3, 1, 1)$, and $\mathbf{w} = (4, -7, 3)$. Determine whether $\mathbf{w}$ belongs to $\mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2)$.

We have to check if there exist $r_1, r_2 \in \mathbb{R}$ such that $\mathbf{w} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2$. This vector equation is equivalent to a system of linear equations:

$$\begin{cases} 4 = r_1 + 3r_2 \\ -7 = 2r_1 + r_2 \\ 3 = 0r_1 + r_2 \end{cases} \iff \begin{cases} r_1 = -5 \\ r_2 = 3 \end{cases}$$

Thus $\mathbf{w} = -5\mathbf{v}_1 + 3\mathbf{v}_2$ is in $\mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2)$.

Problem Let $\mathbf{v}_1 = (2,5)$ and $\mathbf{v}_2 = (1,3)$. Show that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a spanning set for $\mathbb{R}^2$.

Take any vector $\mathbf{w} = (a, b) \in \mathbb{R}^2$. We have to check that there exist $r_1, r_2 \in \mathbb{R}$ such that

$$\mathbf{w} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 \iff \begin{cases} 2r_1 + r_2 = a \\ 5r_1 + 3r_2 = b \end{cases}$$

Coefficient matrix: $C = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$. $\det C = 1 \neq 0$.

Since the matrix C is invertible, the system has a unique solution for any a and b.
Thus $\mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2) = \mathbb{R}^2$.

Problem Let $\mathbf{v}_1 = (2,5)$ and $\mathbf{v}_2 = (1,3)$. Show that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a spanning set for $\mathbb{R}^2$.

Alternative solution: First let us show that vectors $\mathbf{e}_1 = (1,0)$ and $\mathbf{e}_2 = (0,1)$ belong to $\mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2)$.

$$\mathbf{e}_1 = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 \iff \begin{cases} 2r_1 + r_2 = 1 \\ 5r_1 + 3r_2 = 0 \end{cases} \iff \begin{cases} r_1 = 3 \\ r_2 = -5 \end{cases}$$

$$\mathbf{e}_2 = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 \iff \begin{cases} 2r_1 + r_2 = 0 \\ 5r_1 + 3r_2 = 1 \end{cases} \iff \begin{cases} r_1 = -1 \\ r_2 = 2 \end{cases}$$

Thus $\mathbf{e}_1 = 3\mathbf{v}_1 - 5\mathbf{v}_2$ and $\mathbf{e}_2 = -\mathbf{v}_1 + 2\mathbf{v}_2$.
Then for any vector $\mathbf{w} = (a,b) \in \mathbb{R}^2$ we have

$$\mathbf{w} = a\mathbf{e}_1 + b\mathbf{e}_2 = a(3\mathbf{v}_1 - 5\mathbf{v}_2) + b(-\mathbf{v}_1 + 2\mathbf{v}_2)$$
$$= (3a - b)\mathbf{v}_1 + (-5a + 2b)\mathbf{v}_2.$$

Problem Let $\mathbf{v}_1 = (2,5)$ and $\mathbf{v}_2 = (1,3)$. Show that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a spanning set for $\mathbb{R}^2$.

Remarks on the alternative solution:

Notice that $\mathbb{R}^2$ is spanned by vectors $\mathbf{e}_1 = (1,0)$ and $\mathbf{e}_2 = (0,1)$ since $(a,b) = a\mathbf{e}_1 + b\mathbf{e}_2$.

This is why we have checked that vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ belong to $\mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2)$. Then

$$\mathbf{e}_1, \mathbf{e}_2 \in \mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2) \implies \mathrm{Span}(\mathbf{e}_1, \mathbf{e}_2) \subset \mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2)$$
$$\implies \mathbb{R}^2 \subset \mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2) \implies \mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2) = \mathbb{R}^2.$$

In general, to show that $\mathrm{Span}(S_1) = \mathrm{Span}(S_2)$, it is enough to check that $S_1 \subset \mathrm{Span}(S_2)$ and $S_2 \subset \mathrm{Span}(S_1)$.

More properties of span

Let S_0 and S be subsets of a vector space V.

- $S_0 \subset S \implies \mathrm{Span}(S_0) \subset \mathrm{Span}(S)$.
- $\mathrm{Span}(S_0) = V$ and $S_0 \subset S \implies \mathrm{Span}(S) = V$.
- If $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_k$ is a spanning set for V and $\mathbf{v}_0$ is a linear combination of vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ then $\mathbf{v}_1, \dots, \mathbf{v}_k$ is also a spanning set for V.

Indeed, if $\mathbf{v}_0 = r_1\mathbf{v}_1 + \cdots + r_k\mathbf{v}_k$, then
$t_0\mathbf{v}_0 + t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k = (t_0 r_1 + t_1)\mathbf{v}_1 + \cdots + (t_0 r_k + t_k)\mathbf{v}_k$.

- $\mathrm{Span}(S_0 \cup \{\mathbf{v}_0\}) = \mathrm{Span}(S_0)$ if and only if $\mathbf{v}_0 \in \mathrm{Span}(S_0)$.

If $\mathbf{v}_0 \in \mathrm{Span}(S_0)$, then $S_0 \cup \mathbf{v}_0 \subset \mathrm{Span}(S_0)$, which implies $\mathrm{Span}(S_0 \cup \{\mathbf{v}_0\}) \subset \mathrm{Span}(S_0)$. On the other hand, $\mathrm{Span}(S_0) \subset \mathrm{Span}(S_0 \cup \{\mathbf{v}_0\})$.