MATH 304

Linear Algebra

# Lecture 17:
# Basis and dimension (continued).
# Rank of a matrix.

# Basis

*Definition.* Let $V$ be a vector space. A linearly independent spanning set for $V$ is called a **basis**.

Equivalently, a nonempty subset $S \subset V$ is a basis for $V$ if any vector $\mathbf{v} \in V$ is *uniquely represented* as a linear combination

$$\mathbf{v} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k,$$

where $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are distinct vectors from $S$ and $r_1, \ldots, r_k \in \mathbb{R}$.

# Dimension

**Theorem 1** Any vector space has a basis.

**Theorem 2** If a vector space $V$ has a finite basis, then all bases for $V$ are finite and have the same number of elements.

*Definition.* The **dimension** of a vector space $V$, denoted $\dim V$, is the number of elements in any of its bases.

*Examples.* • $\dim \mathbb{R}^n = n$

- $\mathcal{M}_{m,n}(\mathbb{R})$: the space of $m\times n$ matrices; $\dim \mathcal{M}_{m,n} = mn$
- $\mathcal{P}_n$: polynomials of degree less than $n$; $\dim \mathcal{P}_n = n$
- $\mathcal{P}$: the space of all polynomials; $\dim \mathcal{P} = \infty$
- $\{\mathbf{0}\}$: the trivial vector space; $\dim \{\mathbf{0}\} = 0$

## How to find a basis?

**Theorem** Let $V$ be a vector space. Then
**(i)** any spanning set for $V$ contains a basis;
**(ii)** any linearly independent subset of $V$ is contained in a basis.

*Approach 1.* Given a spanning set for the vector space, reduce this set to a basis.

*Approach 2.* Given a linearly independent set, extend this set to a basis.

*Approach 2a.* Given a spanning set $S_1$ and a linearly independent set $S_2$, extend the set $S_2$ to a basis adding vectors from the set $S_1$.

Vectors $\mathbf{v}_1 = (0, 1, 0)$ and $\mathbf{v}_2 = (-2, 0, 1)$ are linearly independent.

**Problem.** Extend the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ to a basis for $\mathbb{R}^3$.

Our task is to find a vector $\mathbf{v}_3$ that is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. Then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ will be a basis for $\mathbb{R}^3$.

Since vectors $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, and $\mathbf{e}_3 = (0, 0, 1)$ form a spanning set for $\mathbb{R}^3$, at least one of them can be chosen as $\mathbf{v}_3$.

One can check that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_1\}$ and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_3\}$ are two bases for $\mathbb{R}^3$:

$$\begin{vmatrix} 0 & -2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 1 \neq 0, \qquad \begin{vmatrix} 0 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 2 \neq 0.$$

**Problem.** Find a basis for the vector space $V$ spanned by vectors $\mathbf{w}_1 = (1,1,0)$, $\mathbf{w}_2 = (0,1,1)$, $\mathbf{w}_3 = (2,3,1)$, and $\mathbf{w}_4 = (1,1,1)$.

To pare this spanning set, we need to find a relation of the form $r_1\mathbf{w}_1+r_2\mathbf{w}_2+r_3\mathbf{w}_3+r_4\mathbf{w}_4 = \mathbf{0}$, where $r_i \in \mathbb{R}$ are not all equal to zero. Equivalently,

$$\begin{pmatrix} 1 & 0 & 2 & 1 \\ 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

To solve this system of linear equations for $r_1, r_2, r_3, r_4$, we apply row reduction.

$$\begin{pmatrix} 1 & 0 & 2 & 1 \\ 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} \color{red}{1} & 0 & 2 & 1 \\ 0 & \color{red}{1} & 1 & 0 \\ 0 & 0 & 0 & \color{red}{1} \end{pmatrix}$$

$$\to \begin{pmatrix} \color{red}{1} & 0 & 2 & 0 \\ 0 & \color{red}{1} & 1 & 0 \\ 0 & 0 & 0 & \color{red}{1} \end{pmatrix} \qquad \text{(reduced row echelon form)}$$

$$\begin{cases} r_1 + 2r_3 = 0 \\ r_2 + r_3 = 0 \\ r_4 = 0 \end{cases} \iff \begin{cases} r_1 = -2r_3 \\ r_2 = -r_3 \\ r_4 = 0 \end{cases}$$

General solution: $(r_1, r_2, r_3, r_4) = (-2t, -t, t, 0)$, $t \in \mathbb{R}$.

Particular solution: $(r_1, r_2, r_3, r_4) = (2, 1, -1, 0)$.

**Problem.** Find a basis for the vector space $V$ spanned by vectors $\mathbf{w}_1 = (1, 1, 0)$, $\mathbf{w}_2 = (0, 1, 1)$, $\mathbf{w}_3 = (2, 3, 1)$, and $\mathbf{w}_4 = (1, 1, 1)$.

We have obtained that $2\mathbf{w}_1 + \mathbf{w}_2 - \mathbf{w}_3 = \mathbf{0}$. Hence any of vectors $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ can be dropped. For instance, $V = \mathrm{Span}(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_4)$.

Let us check whether vectors $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_4$ are linearly independent:

$$\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 \neq 0.$$

They are!!! It follows that $V = \mathbb{R}^3$ and $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_4\}$ is a basis for $V$.

# Row space of a matrix

*Definition.* The **row space** of an $m \times n$ matrix $A$ is the subspace of $\mathbb{R}^n$ spanned by rows of $A$.
The dimension of the row space is called the **rank** of the matrix $A$.

**Theorem 1** The rank of a matrix $A$ is the maximal number of linearly independent rows in $A$.

**Theorem 2** Elementary row operations do not change the row space of a matrix.

**Theorem 3** If a matrix $A$ is in row echelon form, then the nonzero rows of $A$ are linearly independent.

**Corollary** The rank of a matrix is equal to the number of nonzero rows in its row echelon form.

**Theorem** Elementary row operations do not change the row space of a matrix.

*Proof:* Suppose that $A$ and $B$ are $m \times n$ matrices such that $B$ is obtained from $A$ by an elementary row operation. Let $\mathbf{a}_1, \ldots, \mathbf{a}_m$ be the rows of $A$ and $\mathbf{b}_1, \ldots, \mathbf{b}_m$ be the rows of $B$. We have to show that $\mathrm{Span}(\mathbf{a}_1, \ldots, \mathbf{a}_m) = \mathrm{Span}(\mathbf{b}_1, \ldots, \mathbf{b}_m)$.

Observe that any row $\mathbf{b}_i$ of $B$ belongs to $\mathrm{Span}(\mathbf{a}_1, \ldots, \mathbf{a}_m)$. Indeed, either $\mathbf{b}_i = \mathbf{a}_j$ for some $1 \leq j \leq m$, or $\mathbf{b}_i = r\mathbf{a}_i$ for some scalar $r \neq 0$, or $\mathbf{b}_i = \mathbf{a}_i + r\mathbf{a}_j$ for some $j \neq i$ and $r \in \mathbb{R}$.

It follows that $\mathrm{Span}(\mathbf{b}_1, \ldots, \mathbf{b}_m) \subset \mathrm{Span}(\mathbf{a}_1, \ldots, \mathbf{a}_m)$.

Now the matrix $A$ can also be obtained from $B$ by an elementary row operation. By the above,

$$\mathrm{Span}(\mathbf{a}_1, \ldots, \mathbf{a}_m) \subset \mathrm{Span}(\mathbf{b}_1, \ldots, \mathbf{b}_m).$$

**Problem.** Find the rank of the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Elementary row operations do not change the row space. Let us convert $A$ to row echelon form:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} \color{red}{1} & 1 & 0 \\ 0 & \color{red}{1} & 1 \\ 0 & 0 & \color{red}{1} \\ 0 & 0 & 0 \end{pmatrix}$$

Vectors $(1, 1, 0)$, $(0, 1, 1)$, and $(0, 0, 1)$ form a basis for the row space of $A$. Thus the rank of $A$ is 3.

It follows that the row space of $A$ is the entire space $\mathbb{R}^3$.

**Problem.** Find a basis for the vector space $V$ spanned by vectors $\mathbf{w}_1 = (1, 1, 0)$, $\mathbf{w}_2 = (0, 1, 1)$, $\mathbf{w}_3 = (2, 3, 1)$, and $\mathbf{w}_4 = (1, 1, 1)$.

The vector space $V$ is the row space of a matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

According to the solution of the previous problem, vectors $(1, 1, 0)$, $(0, 1, 1)$, and $(0, 0, 1)$ form a basis for $V$.

# Column space of a matrix

*Definition.* The **column space** of an $m \times n$ matrix $A$ is the subspace of $\mathbb{R}^m$ spanned by columns of $A$.

**Theorem 1** The column space of a matrix $A$ coincides with the row space of the transpose matrix $A^T$.

**Theorem 2** Elementary column operations do not change the column space of a matrix.

**Theorem 3** Elementary row operations do not change the dimension of the column space of a matrix (although they can change the column space).

**Theorem 4** For any matrix, the row space and the column space have the same dimension.

**Problem.** Find a basis for the column space of the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

The column space of $A$ coincides with the row space of $A^T$. To find a basis, we convert $A^T$ to row echelon form:

$$A^T = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Vectors $(1, 0, 2, 1)$, $(0, 1, 1, 0)$, and $(0, 0, 0, 1)$ form a basis for the column space of $A$.

**Problem.** Find a basis for the column space of the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

*Alternative solution:* We already know from a previous problem that the rank of $A$ is 3. It follows that the columns of $A$ are linearly independent. Therefore these columns form a basis for the column space.