

MATH 304

Linear Algebra

Lecture 22:

General linear equations (continued).

Matrix transformations.

Matrix of a linear transformation.

Range and kernel

Definition. Given vector spaces V and W , a mapping $L : V \rightarrow W$ is **linear** if

$$L(\mathbf{x} + \mathbf{y}) = L(\mathbf{x}) + L(\mathbf{y}),$$

$$L(r\mathbf{x}) = rL(\mathbf{x})$$

for any $\mathbf{x}, \mathbf{y} \in V$ and $r \in \mathbb{R}$.

Definition. The **range** (or **image**) of L is the set of all vectors $\mathbf{w} \in W$ such that $\mathbf{w} = L(\mathbf{v})$ for some $\mathbf{v} \in V$. The range of L is denoted $L(V)$.

The **kernel** of L , denoted $\ker L$, is the set of all vectors $\mathbf{v} \in V$ such that $L(\mathbf{v}) = \mathbf{0}$.

General linear equations

Definition. A **linear equation** is an equation of the form

$$L(\mathbf{x}) = \mathbf{b},$$

where $L : V \rightarrow W$ is a linear mapping, \mathbf{b} is a given vector from W , and \mathbf{x} is an unknown vector from V .

The range of L is the set of all vectors $\mathbf{b} \in W$ such that the equation $L(\mathbf{x}) = \mathbf{b}$ has a solution.

The kernel of L is the solution set of the **homogeneous** linear equation $L(\mathbf{x}) = \mathbf{0}$.

Theorem If the linear equation $L(\mathbf{x}) = \mathbf{b}$ is solvable and $\dim \ker L < \infty$, then the general solution is

$$\mathbf{x}_0 + t_1 \mathbf{v}_1 + \cdots + t_k \mathbf{v}_k,$$

where \mathbf{x}_0 is a particular solution, $\mathbf{v}_1, \dots, \mathbf{v}_k$ is a basis for the kernel of L , and t_1, \dots, t_k are arbitrary scalars.

Example.
$$\begin{cases} x + y + z = 4, \\ x + 2y = 3. \end{cases}$$

$$L: \mathbb{R}^3 \rightarrow \mathbb{R}^2, \quad L \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Linear equation: $L(\mathbf{x}) = \mathbf{b}$, where $\mathbf{b} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & 2 & 0 & 3 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & -1 & -1 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 2 & 5 \\ 0 & 1 & -1 & -1 \end{array} \right)$$

$$\begin{cases} x + 2z = 5 \\ y - z = -1 \end{cases} \iff \begin{cases} x = 5 - 2z \\ y = -1 + z \end{cases}$$

$$(x, y, z) = (5 - 2t, -1 + t, t) = (5, -1, 0) + t(-2, 1, 1).$$

Example. $u'''(x) - 2u''(x) + u'(x) = e^{2x}$.

Linear operator $L : C^3(\mathbb{R}) \rightarrow C(\mathbb{R})$,

$$Lu = u''' - 2u'' + u'.$$

Linear equation: $Lu = b$, where $b(x) = e^{2x}$.

We know from the previous lecture that functions xe^x , e^x and 1 form a basis for the kernel of L . It remains to find a particular solution.

$$L(e^{2x}) = 8e^{2x} - 2(4e^{2x}) + 2e^{2x} = 2e^{2x}.$$

Since L is a linear operator, $L\left(\frac{1}{2}e^{2x}\right) = e^{2x}$.

Particular solution: $u_0(x) = \frac{1}{2}e^{2x}$.

Thus the general solution is

$$u(x) = \frac{1}{2}e^{2x} + t_1xe^x + t_2e^x + t_3.$$

Matrix transformations

Any $m \times n$ matrix A gives rise to a transformation $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$ given by $L(\mathbf{x}) = A\mathbf{x}$, where $\mathbf{x} \in \mathbb{R}^n$ and $L(\mathbf{x}) \in \mathbb{R}^m$ are regarded as column vectors. This transformation is **linear**.

Example.
$$L \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 4 & 7 \\ 0 & 5 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Let $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, $\mathbf{e}_3 = (0, 0, 1)$ be the standard basis for \mathbb{R}^3 . We have that $L(\mathbf{e}_1) = (1, 3, 0)$, $L(\mathbf{e}_2) = (0, 4, 5)$, $L(\mathbf{e}_3) = (2, 7, 8)$. Thus $L(\mathbf{e}_1)$, $L(\mathbf{e}_2)$, $L(\mathbf{e}_3)$ are columns of the matrix.

Problem. Find a linear mapping $L : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ such that $L(\mathbf{e}_1) = (1, 1)$, $L(\mathbf{e}_2) = (0, -2)$, $L(\mathbf{e}_3) = (3, 0)$, where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is the standard basis for \mathbb{R}^3 .

$$\begin{aligned}L(x, y, z) &= L(x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3) \\ &= xL(\mathbf{e}_1) + yL(\mathbf{e}_2) + zL(\mathbf{e}_3) \\ &= x(1, 1) + y(0, -2) + z(3, 0) = (x + 3z, x - 2y)\end{aligned}$$

$$L(x, y, z) = \begin{pmatrix} x + 3z \\ x - 2y \end{pmatrix} = \begin{pmatrix} 1 & 0 & 3 \\ 1 & -2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Columns of the matrix are vectors $L(\mathbf{e}_1), L(\mathbf{e}_2), L(\mathbf{e}_3)$.

Theorem Suppose $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear map. Then there exists an $m \times n$ matrix A such that $L(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. Columns of A are vectors $L(\mathbf{e}_1), L(\mathbf{e}_2), \dots, L(\mathbf{e}_n)$, where $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ is the standard basis for \mathbb{R}^n .

$$\mathbf{y} = A\mathbf{x} \iff \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

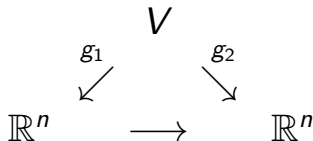
$$\iff \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \dots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

Change of coordinates (revisited)

Let V be a vector space.

Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ be a basis for V and $g_1 : V \rightarrow \mathbb{R}^n$ be the coordinate mapping corresponding to this basis.

Let $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ be another basis for V and $g_2 : V \rightarrow \mathbb{R}^n$ be the coordinate mapping corresponding to this basis.



The composition $g_2 \circ g_1^{-1}$ is a linear mapping of \mathbb{R}^n to itself. Hence it's represented as $\mathbf{x} \mapsto U\mathbf{x}$, where U is an $n \times n$ matrix.

U is called the **transition matrix** from $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ to $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$. Columns of U are coordinates of the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ with respect to the basis $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$.

Matrix of a linear transformation

Let V, W be vector spaces and $f : V \rightarrow W$ be a linear map.

Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ be a basis for V and $g_1 : V \rightarrow \mathbb{R}^n$ be the coordinate mapping corresponding to this basis.

Let $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m$ be a basis for W and $g_2 : W \rightarrow \mathbb{R}^m$ be the coordinate mapping corresponding to this basis.

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ g_1 \downarrow & & \downarrow g_2 \\ \mathbb{R}^n & \longrightarrow & \mathbb{R}^m \end{array}$$

The composition $g_2 \circ f \circ g_1^{-1}$ is a linear mapping of \mathbb{R}^n to \mathbb{R}^m . Hence it's represented as $\mathbf{x} \mapsto A\mathbf{x}$, where A is an $m \times n$ matrix.

A is called the **matrix of f** with respect to bases $\mathbf{v}_1, \dots, \mathbf{v}_n$ and $\mathbf{w}_1, \dots, \mathbf{w}_m$. Columns of A are coordinates of vectors $f(\mathbf{v}_1), \dots, f(\mathbf{v}_n)$ with respect to the basis $\mathbf{w}_1, \dots, \mathbf{w}_m$.

Examples. • $D : \mathcal{P}_3 \rightarrow \mathcal{P}_2$, $(Dp)(x) = p'(x)$.

Let A_D be the matrix of D with respect to the bases $1, x, x^2$ and $1, x$. Columns of A_D are coordinates of polynomials $D1, Dx, Dx^2$ w.r.t. the basis $1, x$.

$$D1 = 0, Dx = 1, Dx^2 = 2x \implies A_D = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

• $L : \mathcal{P}_3 \rightarrow \mathcal{P}_3$, $(Lp)(x) = p(x + 1)$.

Let A_L be the matrix of L w.r.t. the basis $1, x, x^2$.

$$L1 = 1, Lx = 1 + x, Lx^2 = (x + 1)^2 = 1 + 2x + x^2.$$

$$\implies A_L = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$